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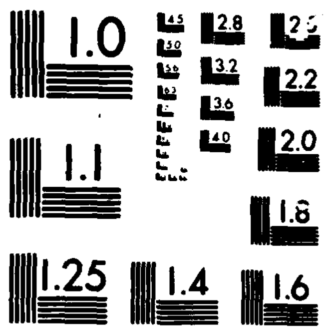

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A SEMI-ANALYTICAL APPROACH TO THE INTEGRAL EQUATION (IN TERMS OF THE ACCELERATION POTENTIAL) FOR THE LINEARIZED SUBSONIC, OSCILLATORY FLOW OVER AN AIRFOIL

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March 1986

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This technical report has been reviewed and is approved for publication.

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The kernel of the integral equation which describes the oscillatory flow over an airfoil has severe singularities. The mathematical questions that arise here are usually circumvented by the use of physical concepts; one concentrates, for instance, the pressures into lines, or even points and evaluates the upwash field by means of the Biot Savart law. The upwash so expressed in terms of the parameters that describe the pressure field is then matched at selected "control" points with the upwash prescribed by the boundary conditions. One thus obtains a finite linear system of equations for the parameters describing the pressure distribution. The selection of control points contains an element of uncertainty, because the upwash field generated by a pressure distribution of the assumed character is highly singular. In the present approach the pressure distribution is again represented by a finite number of parameters and one ultimately solves a linear system of equations. But for the pressure a much smoother representation is chosen (piece-wise linear continuous functions). The upwash is then computed directly from the integral equation. Matching

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between the upwash distribution due to the unknown pressure distribution and that given by the boundary conditions is carried out in the average over "upwash areas." Such a procedure requires repeated integrations. The upwash at a distance from a pressure element is a smooth function and numerical integration is economical. But for close points the functions change too rapidly for a numerical approach. Analytical integrations are possible if one uses a development of the kernel of the integral equation with respect to the frequency which is due to Ueda, and by a suitable choice of the functions by which the pressure is represented. The necessary analytical discussions are carried out in considerable detail. One nontrivial aspect is the interpretation of the singularities which occur in the integral equation.

PREFACE

This report has been written under Contract F33615-83-K-3207 entitled "Investigation of Techniques for Computing Steady and Unsteady Transonic and Supersonic Flows" to the University of Dayton for the Aeroelastic Group, Analysis and Optimization Branch, Structures and Dynamics Division (AFWAL/FIBRC), Air Force Wright Aeronautical Laboratories under Project 2304, Task 2304N1, and Program Element 61102F.

The work was performed during the period March 1983 to September 1985. Dr. Karl G. Guderley of the University of Dayton Research Institute was Principal Investigator. Dr. Charles L. Keller, AFWAL/FIBRC, (513) 255-7384, Wright-Patterson Air Force Base, Ohio, was Program Manager.

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SECTION I
INTRODUCTION

The present report gives a version of linearized airfoil theory for subsonic oscillatory flows which avoids the frequently-used idea of concentrating the pressures into lines or points. As usual, the linearized flow differential equation is solved by means of fundamental solutions. The boundary conditions at the planform then are expressed by an integral equation. In this report, the acceleration potential is used. It has the advantage of giving directly, without differentiations, the pressure distributions. Consequently, the Kutta condition expresses itself very simply by the postulate that there is no pressure difference between the upper and lower side of the wing. However, the singularities which occur in the governing integral equation are very strong and this causes conceptual and practical difficulties.

For numerical purposes, one always expresses the pressure distribution in terms of a finite number of parameters. In the vortex lattice method (which can be viewed as a discretization of the integral equation formulation) the pressures are concentrated into lines along which the force per unit of length is piece-wise constant. For such lines (and the pertinent trailing vortices), the upwash can be computed by the Biot-Savart law. Dowell and Ueda (Refs. 1,2), on the other hand, concentrate the pressures into "pressure points;" although for points of the wing surface lying in the wake of the pressure points, this concept must be modified. Dowell and Ueda do this without further explanation by reference to Mangler's work (Ref. 3). (It seems to me that Mangler's approach is applicable only if for the upwash at points of the wake one replaces the point force by a line force, or alternatively if one averages the upwash along a line in the wake.) To express the boundary conditions at the plan form, one usually matches at a sufficient number of "control" points, the upwash expressed in terms of the pressure parameters and the upwash given by the boundary conditions. One then obtains a linear system (with a full matrix) from which the pressure parameters are determined.

The upwash field obtained from the pressure field is by no means smooth. The choice of control points therefore introduces an element of arbitrariness. Nevertheless, these methods are successful, although perhaps not overly accurate.

In the present study, the author reduces the arbitrariness caused by the concept of pressure points or pressure lines and by the choice of control points. (Some arbitrariness is inherent in any discretization.) Within surface elements (preferably triangles), an expression closely related to the pressure is approximated by linear functions. The pressure is continuous as one proceeds from one element to its neighbors. In a cruder form, one may also use constant pressure elements. (Even in the latter case, the wake has only logarithmic singularities if no side of the (polygonal) element is parallel to the wake streamlines.) The arbitrariness due to choice of the control points is avoided by matching the upwash velocities in the *average over control* areas (which include lines along which the flow field is singular). While this is conceptually satisfactory, it complicates the procedure since it requires further integrations. For control areas at a distance from the pressure areas and their wakes, the functions to be integrated are smooth. The integration can then be carried out numerically, for instance, by (low order) Gaussian integration. The results are rather close to those obtained with the idea of pressure and control points. (This is the reason for the success of the method of Ueda and Dowell.) In the vicinity of the pressure element, or of its wake, it is preferable to determine the part of the upwash in which singularities occur analytically. One then uses a mixed numerical analytical method. This works well for control areas close to the wake of the pressure element but at some distance from the pressure area.

If the control element is close to the pressure element, a corresponding splitting of the upwash is possible in principle. But then even the "smooth" function to be integrated proves to be rather intractable for numerical integration methods. Fortunately, all necessary integrations can be carried out analytically in terms of elementary transcedental functions. This is possible because of several factors: (1) Ueda has provided a development of the kernel of the integral equation with respect to the reduced frequency. If the elements are close to each other, only the terms of the lowest order, which are relatively simple, are of importance. (2) If the wing surface is subdivided into triangles, then it is possible to represent the pressure distribution (roughly speaking) by linear functions which are continuous, and at the same time, allow one to carry out the necessary integrations. (3) The choice of the coordinate systems is important. To obtain the upwash at a point (x,y) one uses coordinates which have this point as origin. One of the two integrations over the pressure element is then trivial. For the integration over the control elements, one uses as origins the corner points of the pressure elements with a similar effect.

The basic equations and the formulae of Ueda (with some extension) are derived in Appendix A and B. The integral equation directly written for the plane of the wing is meaningless. The author found it preferable to go back to the original meaning and include the limiting process in which one approaches the plane of the wing from above or below. Mostly, but not always, one obtains the same result as in the usual less cautious approach. Integration formulae needed for the determination of the upwash and later for the averaging over the control area are first derived for distant points close to the wake and later for cases where pressure elements and control elements are close to each other.

The complexity of the resulting formulae has given the author some uneasiness because of the danger of errors. The analytical procedure is shown in sufficient detail so that the reader will be able to check the results himself.

A survey of essential ideas and a listing of the formulae needed in the computation is found in Section VIII.

SECTION II
BASIC EQUATIONS

Let $\tilde{x},\tilde{y},\tilde{z}$ be a system of Cartesian coordinates. We consider the subsonic oscillatory flow over a wing in a linearized approach. The plane of the wing is given by $\tilde{z} = 0$. Coordinates within the wing planform corresponding to $\tilde{x}$ and $\tilde{y}$ are $\tilde{\xi}$ and $\tilde{\eta}$ respectively. The free stream velocity is U, the free stream Mach number M, and the free stream density ρ. All lengths are made dimensionless with a characteristic length L. The time dependence of the oscillatory motion is given by the factor $\exp(i\nu t)$, where t is the time. The dimensionless frequency is $k = \nu L/U$. The deviation of the pressure from the free stream pressure is made dimensionless with $\rho U^2/2$ and denoted by p. The dimensionless pressure difference between the lower and upper sides of the wing is denoted by Δp. The dimensionless upwash at a point $\tilde{x},\tilde{y},\tilde{z}$ is denoted by $\tilde{w}$. The upwash at a point $\tilde{x},\tilde{y},\tilde{z}$ expressed in terms of the pressure difference is given by

$$\tilde{w}(\tilde{x},\tilde{y},\tilde{z}) = (8\pi)^{-1} \frac{\partial}{\partial \tilde{z}} [\tilde{z} \iint_{\tilde{A}} \Delta\tilde{p}(\tilde{\xi},\tilde{\eta})\tilde{K}(\tilde{\xi}-\tilde{x},\tilde{r},\tilde{k})d\tilde{\xi}d\tilde{\eta}] \tag{1}$$

The region of integration is the wing area $\tilde{A}$. K is given by the classical formula

$$\tilde{K}(\tilde{\xi}-\tilde{x},\tilde{r},k) = \exp(ik(\tilde{\xi}-\tilde{x}))\left[\frac{M\exp(ik\tilde{V})}{\tilde{R}(\tilde{V}^2+\tilde{r}^2)^{1/2}} + \int_{-\infty}^{\tilde{V}} \frac{\exp(ikv)}{(\tilde{V}^2+\tilde{r}^2)^{3/2}}\,d\tilde{v}\right] \tag{2}$$

with

$$\begin{aligned} \tilde{r}^2 &= (\tilde{\eta}-\tilde{y})^2 + \tilde{z}^2 \\ \tilde{R}^2 &= [(\tilde{x}-\tilde{\xi})^2 + \beta^2\tilde{r}^2] \\ \tilde{V} &= (1-M^2)^{-1}(-(\tilde{\xi}-\tilde{x})-M\tilde{R}) \end{aligned} \tag{3}$$

$$\beta = (1-M^2)^{1/2} \qquad (3)\text{ cont'd.}$$

A derivation is found in Appendix A. We introduce the Prandtl Glauert transformation

$$x = \tilde{x}$$

$$y = \beta\tilde{y}$$

$$z = \beta\tilde{z}$$

$$\xi = \tilde{\xi}$$

$$\eta = \beta\tilde{\eta}$$

$$\tilde{w}(x,\beta^{-1}y,\beta^{-1}z) = w(x,y,z)$$

$$\Delta\tilde{p}(\xi,\beta^{-1}\eta) = \Delta p(\xi,\eta) \qquad (4)$$

$$\beta^{-1}\tilde{K}(\xi-x,\beta^{-1}r,k) = K(\xi-x,r,k)$$

with

$$r^2 = (\eta-y)^2 + z^2$$

Then

$$w(x,y,z) = (8\pi)^{-1} \frac{\partial}{\partial z}(z \iint_A \Delta p(\xi,\eta)K(\xi-x,r,k)d\xi d\eta) \qquad (5)$$

The power series development with respect to the reduced frequency k of K, Eq. (2), has been derived by Ueda. (The result is by no means obvious, because the integrals which arise if one develops the integrand of Eq. (2) with respect to k do not converge.) In Appendix B these formulae have been rederived. Simplifications which are rather important for the present approach arise if one partially combines the two terms on the right of Eq. (2). The form used here is given in Eqs. (B.63, B.64, and B.65).

The pressure difference Δp is found from an integral equation, obtained from Eq. (5) by making the limiting process $z \to 0$ on the right and by substituting on the left the upwash which, of course, is known over the wing. Because of the singularity of K the evaluation of the right-hand side involves a limiting process which is not trivial except for points (x,y) at some distance from the point (ξ,η) and its wake. It is true one obtains ultimately $\lim_{z \to 0} \frac{\partial}{\partial z}(z \iint ..) = \lim_{z \to 0} \iint ...$, except for one important exception which justifies the cautious approach taken in this report. A reference to the work of Mangler (Ref. 4) does not seem to be sufficient.

In the approach taken here the wing planform is divided into elements and within each element the (unknown) pressure is approximated by a linear combination of shape functions. The coefficients by which the shape functions are multiplied are the pressure parameters so far unknown. The upwash distribution is expressed in terms of these parameters. Usually a system of equations for these parameters is obtained by equating the upwash at certain points, called upwash points or control points, with the upwash given by the boundary conditions. The upwash pertaining to the chosen pressure distribution is by no means smooth. At the element boundaries and certain lines of the wake pertaining to a pressure element it goes logarithmically to infinity; for constant pressure elements it even behaves as d^{-1} where d is the distance from the boundary. One chooses the control points at a distance from these singularities, but in any case, the results will be rather inaccurate (except of course if the points (x,y) are at some distance from the point (ξ,η) and its wake).

In this report we equate the integrals over the given downwash over certain areas of the wing with the same integral over the downwash expressed in terms of the parameters for the pressures. These areas will be called control areas.

As the work progressed it became more and more apparent that analytical integrations would play an important role. The kernel of the integral equation is accompanied by a factor

$\exp ik(\xi - x)$. We set

$$\Delta\bar{p}(\xi,\eta) = \Delta p(\xi,\eta)\exp(ik\xi) \qquad (6)$$

This is one of the steps which make an analytical integration feasible. Moreover, we introduce a weight function exp(ikx) in the integrals for the upwash. Since the reduced frequency is not large, $\exp(ik\xi)$ and exp(ikx) vary only by a small amount within an element; or even within the pressure and control areas.

During the course of the work the author's idea about the choice of the pressure and of the control elements have undergone changes. First he had a subdivision of the planform in mind which is suggested by the vortex-lattice method, and which has been used in the work of Ueda and Dowell (Fig. 1). Then he realized that for such trapezoidal elements it is not possible to find elemental pressure shape functions which satisfy the two requirements that the pressures be continuous as one passes from one element to the next and that the resulting integrals can be integrated analytically. Triangular elements with linear shape functions for $\Delta p(\xi,\eta)$ are preferable. Such elements can be obtained by drawing one diagonal into the trapezoids of Fig. 1 (see Fig. 2).

In the triangles in Fig. 2 one side is parallel to the x-axis (to the direction of the wake streamlnes). Consider a single element and assume that $\Delta\bar{p}(\xi,\eta)$ is constant (Fig. 3a). Then one obtains at the wake boundary $\eta = \eta_3$ a singularity in the upwash as $(\eta_3 - y)^{-1}$, at the element boundary and at $\eta = \eta_1$ a singularity as $\log(\eta_1 - y)$. In contrast, if none of the sides of the triangle is parallel to the x-axis, then one obtains in the wake three singular lines $\eta = \eta_1$, $\eta = \eta_2$, and $\eta = \eta_3$ but only with logarithmic singularities. In other words, one obtains a smoother upwash although infinities are still present. This is of particular interest if one works with elements of constant $\Delta\bar{p}$. If the pressure elements have sides parallel to the x-axis, then the control elements must overlap lines of the wake pertaining to the x-axis, otherwise integrals over the upwash will be infinite. For pressure elements with no side parallel to the x-axis this is not necessary, as the logarithmic infinities give a contribution which remains finite after the integration. Incidentally, for constant pressure elements one can use for the elemental areas quadrangles as well as triangles. Such an elemental subdivision is shown in Fig. 4. (Special measures may be needed at the wing tips and at the center line.) A subdivision in triangles is obtained by drawing into the quadrangles the diagonal which is not parallel to the x-axis. Originally, the author thought it desirable to choose the quadrangles or the triangles so that their corners lie on lines of constant y but actually the integration formulae are of a nature that a proliferation of singular lines in the wake causes no additional computational labor.

The numerical work turns out to be quite complicated. It would be greatly simplified if one could use a subdivision into elements which possess a repetitive pattern. For wing plan forms with parallel leading and trailing edges this can easily be achieved (Fig. 5). Here one has only two types of elements (see Fig. 5a) and it suffices if one establishes only once for each type the necessary integrals over the combinations of one pressure element and close upwash elements. If the trailing edge is not parallel to the leading edge and one uses these subdivisions, then one must admit exceptional elements at the trailing edge (see Fig. 6).

For straight leading and trailing edges one can obtain a subdivision which at least has self-similarity (Fig. 7). Let a and b be the chords of the wing at its root and at the tip, respectively. The net of element boundaries to be drawn is self-similar with respect to the point of intersection of leading and trailing edges (point 0). First we choose the cornerpoints of the net that lie on the leading edge. The distances from point 0 are chosen in such a manner that each point is mapped into the next one closer to 0 if one multiplies the scale of the figure by a factor $r < 1$. If there are $n + 1$ points along the leading edge (including the points at the wing root and at the wing tip), then one needs n such transformations to transform the wing root a into the wing tip b. Therefore,

$$r = (b/a)^{1/n}$$

If S is the wing span (measured in the y-direction) then point 0 lies at a distance

$$S_1 = S\ a(a-b)^{-1}$$

The distance of the m^{th} point along the leading edge from point 0, (if one counts from the root outward and assigns m = 0 to the root) is then

$$S_{1m} = S_1 r^n$$

It gives some simplification if the sides of the elements arrange themselves in straight lines. The individual triangular elements are then embedded in larger triangles. Such a net is obtained in the following manner. First one chooses one of these larger triangles with two corners at the points of the leading edge determined above and one corner at the trailing edge. Next one draws straight lines parallel to the sides of these triangles through all those points of the leading edge (including some outside of the wing). This divides the plan form into quadrangles. Because of the choice of the initial points at the leading edge one set of diagonals in these quadrangles form straight lines through point 0. By drawing these diagonals one obtains the desires self-similar net. In general, the other diagonals will not form a straight line and, therefore, the cornerpoints lying on a tract of these diagonals will not form a line y = const. Of course if one of the sides of the original large triangle is line y = const, then all triangles will have such a side. For elements of constant $\Delta\bar{p}$ this complicates the choice of upwash areas. This does not happen for elements in which $\Delta\bar{p}$ is linear.

The parameter describing the pressure distribution are the values of $\Delta\bar{p}(\xi,\eta) = \Delta p(\xi,\eta)\exp(i\kappa\xi)$ at the corners of the grid formed by element boundaries. One parameter therefore generates pressure distributions in all elements that contain the corner which pertains to the parameter whose influence one wants to compute. The region covered by the elements pertaining to a certain pressure parameter will be called the pressure area. In the interior such an area consists of six triangular elements (see Fig. 8). The integration of the upwash (with a weight function exp(ikx) is carried out over "upwash areas." They are identical with the pressure areas. This ensures that the number of equations equals the number of unknowns. The elements of the matrix of the system for the determination of the pressure distribution are given by the upwash due to the pressure in an individual pressure area (with the value 1 assigned to the pertinent pressure parameter) integrated over the upwash areas. (This is, of course, nothing new, the idea is inherent in all methods.) Each element belongs to several pressure and upwash areas. To each triangular element belong three "elemental" pressure distributions. The primary task is the determination of the upwash due to the elemental pressure distribution integrated over the different elements.

For elements with constant $\Delta\bar{p}$, in which no side is parallel to the x-axis, the elements themselves can be used as pressure and upwash areas. If element sides are parallel to the x-axis and $\Delta\bar{p}$ is constant, then the elements still constitute the pressure areas, but the upwash areas must be chosen so that one has an overlap of the singular lines within the wake; this is necessary in order to avoid infinities in the integrated upwash.

Let N be the number of pressure parameters. It is the number of corners in the grid at which the pressure is unknown, i.e., the number of grid corners except for those at the trailing edge where the pressure difference is zero. It is also the number of pressure and of upwash areas. The pressure parameters (each with a pertinent pressure area) and also the upwash areas are numbered from 1 to N. Ultimately, one will generate an N^2 matrix. Further numberings are introduced for the elements and for the elemental pressure distributions. The number of elemental pressure distributions is somewhat smaller than three times the number of elements, because the pressures at the trailing edge are zero. The numberings of the pressure areas and of the upwash areas are the same. The numberings of the elemental pressure distributions and of the upwash elements (with single subscripts) are carried out independently. Let N_E be the number of upwash elements and N_P the number of elemental pressure distributions. Which elemental pressure distributions belong to certain pressure areas and which elements belong to the upwash areas is shown by "housekeeping" matrices $M^{(1)}$ and $M^{(3)}$ of dimension N_P by N and N by N_E, respectively. The elements of these matrices are zero and one. An element $M^{(1)}_{k\ell}$ is one if the elemental pressure distribution with index k pertains to the pressure area with index ℓ. Each row of $M^{(1)}$ contains only a single one, each column a maximum of six. The element $M^{(3)}_{ij}$ is one if the elements with index j belong to the upwash area with index i.

An element $M^{(2)}_{j,k}$ of a third matrix $M^{(2)}$ gives the upwash integrated over the element with index j due to the elemental pressure distribution with index k. The matrix for the system of equations from which the pressure parameters are determined is denoted by M. One has

$$M = M^{(3)}M^{(2)}M^{(1)} \tag{7}$$

The main effort is the determination of the matrix $M^{(2)}$. An element in which the pressure is prescribed will be called $\xi\eta$-element (because of the independent variables for the pressure), and an element over which the upwash is integrated will be called xy-element. In Section III formulae will be developed for elements of $M^{(2)}$ for which the xy-element is at some distance from the $\xi\eta$-element. In subsequent sections xy-elements close to $\xi\eta$-elements will be treated.

SECTION III

EVALUATION OF THE ELEMENTS OF THE MATRIX $M^{(2)}$ FOR xy-ELEMENTS AT A DISTANCE FROM A $\xi\eta$-ELEMENT

The computation is based on Eq. (5). The z-coordinate appears only in the variable r. One has

$$\frac{\partial}{\partial z} = \frac{z}{r}\frac{\partial}{\partial r}$$

If the xy-element is not adjacent to the $\xi\eta$-element or its wake, then the function K is free of singularities. One has

$$\frac{\partial}{\partial z}(zK) = K + (z^2/r)(\partial K/\partial r) \qquad (8)$$

and

$$\lim_{z \to 0}\left(\frac{\partial}{\partial z}(zK)\right) = \lim_{z \to 0} K \qquad (9)$$

One simply replaces $|r|$ by $|\eta - y|$. This is the approach used without restriction in Reference 2.

The specific form of K is found in Appendix B.

$$K = \beta^{-1}\exp(ik(\xi - x))(K_1 + K_2)$$

No wake singularities occur in K_2. The wake singularities occurring in K_1 are displayed in Eq. (B.64a).

We have introduced in Eq. (6)

$$\Delta\bar{p}\ (\xi,\eta) = \exp(ik\xi)\Delta p(\xi,\eta)$$

Eq. (5) will be discretized by postulating

$$\beta 8\pi \iint_{A_i(x,y)} \exp(ikx)w(x,y)dxdy$$
$$= \iint_{A_i(x,y)} \lim_{z\to 0}\{\frac{\partial}{\partial z} z \iint_{A_\ell(\xi,\eta)} \Delta\bar{p}(\xi,\eta)(K_1 + K_2)d\xi d\eta\}dxdy \tag{10}$$

Here $A_i(x,y)$ and $A_\ell(\xi,\eta)$ refer respectively to the i^{th} upwash and the ℓ^{th} pressure area. For a triangular element with corners numbered j, k, ℓ and pressure 1 at corner j, and pressure 0 at corner k and ℓ, the linear pressure shape function is given by

$$\Delta\bar{p}\ (\xi,\eta) = 1 + \frac{\begin{vmatrix}(\xi-\xi_j) & (\xi_k-\xi_\ell)\\(\eta-\eta_j) & (\eta_k-\eta_\ell)\end{vmatrix}}{\begin{vmatrix}(\xi_k-\xi_j) & (\xi_\ell-\xi_j)\\(\eta_k-\eta_j) & (\eta_\ell-\eta_j)\end{vmatrix}} \tag{11}$$

This is easily verified by simple operations on determinants if one sets $\xi = \xi_k$, $\eta = \eta_k$, and $\xi = \xi_\ell$, $\eta = \eta_\ell$. The matrix elements $M^{(2)}_{i\ell}$ is obtained by replacing in Eq. (10) $A_i(x,y)$ and $A_\ell(\xi,\eta)$ respectively by the i^{th} and the ℓ^{th} surface elements.

If the xy-element is not adjacent to the $\xi\eta$-element or its wake, then the integrand of Eq. (10) is analytic in ξ, η, x, and y and the integrations can be carried out with efficient numerical methods. There exist even formulae for Gaussian integration over a triangular area (developed for applications in elasticity).

If the xy-element is close to the wake but at some distance from the $\xi\eta$-element, one proceeds as follows. The function K_1 (see Eq. (B.64a)) is written

$$K_1 = K_3 + K_4$$
$$K_3 = K_{31} + K_{32} \tag{12}$$

with

$$K_{31} = 2\beta^2/r^2$$

$$K_{32} = k^2 \log r \sum_{n=0}^{\infty} \frac{(kr/2\beta)^{2n}}{n!(n+1)!} \tag{13}$$

$$K_4 = -\beta^2[R(R-(\xi-x))]^{-1} - (k^2/2)\log(2k^{-1}(R-(\xi-x))) \sum_{n=0}^{\infty} \frac{(kr/2\beta)^{2n}}{n!(n+1)!} \tag{14}$$

$$r^2 = (\eta-y)^2 + z^2 \tag{15}$$

The function K_2 and K_4 are analytic even in the wake. Their contributions can again be treated by Gaussian integration. $K_3 = K_{31} + K_{32}$ depends upon r only. For the evaluation of this part one divides both the $\xi\eta$-element and the xy-element respectively by a line η=const and a line y=const into smaller triangles (see Fig. 9). The contribution of these smaller triangles are treated separately. $\Delta\bar{p}$ is a linear function of ξ and η

$$\Delta\bar{p}(\xi,\eta) = C_0 + C_1\,\xi + C_2\,\eta \tag{16}$$

Consider a fixed $\xi\eta$ and a fixed xy triangle. Assign the index 1 to the corner opposite to the side parallel to the ξ-axis, and subscripts 2 and 3 to the other corners (proceeding in counterclockwise direction), Figs. 10. We write

$$\Delta\bar{p} = \bar{C}_0 + \bar{C}_1\,(\xi-\xi_1) + \bar{C}_2(\eta-\eta_1) \tag{17}$$

here

$$\bar{C}_0 = C_0 + C_1 \xi_1 + C_2 \eta_1$$

The sides here 1 2 and 1 3 of the triangle are given respectively by

$$\xi - \xi_1 = \frac{\xi_2 - \xi_1}{\eta_2 - \eta_1} (\eta - \eta_1)$$

and

$$\xi - \xi_1 = \frac{\xi_3 - \xi_1}{\eta_3 - \eta_1} (\eta - \eta_1)$$

Then one can carry out the integration over ξ at constant η, and the integrations over x at constant y. Let, for Fig. (10a)

$$f_1(y) = \pm(y-y_1)(x_2-x_3)/(y_2-y_1) \tag{18}$$

$$f_2(\eta) = \pm\left\{[\bar{C}_0 + \bar{C}_2(\eta-\eta_1)](\eta-\eta_1)(\xi_2-\xi_3)/(\eta_2-\eta_1) + (\bar{C}_1/2)(\eta-\eta_1)^2 \frac{(\xi_2-\xi_1)^2-(\xi_3-\xi_1)^2}{(\eta_2-\eta_1)^2}\right\} \tag{19}$$

The upper and lo''er signs hold for Figs. 10(a) and 10(b), respectively. There are three elemental pressure distributions $\Delta\bar{p}(\xi,\eta)$ in each element, and therefore three different sets of constants $\bar{C}_i$, and three *different functions* $f_2(\eta)$. They are quadratic in η.

Let I_{31} be the integral on the right of Eq. (10) with $K_1 + K_2$ replaced by K_{31}. *If the triangles in the* x,y-plane and in the ξ,η-plane have the orientations of Fig. 10(a), one has

$$I_{31} = \int_{y_1}^{y_3} f_1(y) \int_{\eta_1}^{\eta_3} \frac{f_2(\eta)}{(\eta-y)^2 + z^2}\, d\eta\, dy$$

y_1 and y_3 need not be the same as η_1 and η_3 because the x,y-element need not lie in the wake of the $\xi\eta$-element. The inner integration is, of course, carried out at constant y. We write

$$f_2(\eta) = f_2(y) + f_2'(y)(\eta-y) + f_3(\eta,y)(\eta-y)^2 \qquad (20)$$

where

$$f_3(\eta,y) = [f_2(\eta) - f_2(y) - f_2'(y)(\eta-y)]/(\eta-y)^2$$

The function $f_3(\eta,y)$ is analytic at $y = \eta$. Then I_{31} appears in the form

$$I_{31} = I_{311} + I_{312} + I_{313} \qquad (21)$$

where

$$I_{311} = \beta^2 \int_{y_1}^{y_3} f_1(y) f_2(y) \left(\int_{\eta_1}^{\eta_3} \frac{d\eta}{(\eta-y)^2 + z^2} \right) dy \qquad (22)$$

$$I_{312} = \beta^2 \int_{y_1}^{y_3} f_1(y) f_2'(y) \left(\int_{\eta_1}^{\eta_3} \frac{(\eta-y)\,d\eta}{(\eta-y)^2 + z^2} \right) dy \qquad (23)$$

$$I_{313} = \int_{y_1}^{y_3} f_1(y) \int_{\eta_1}^{\eta_3} f_3(\eta,y)\,d\eta - z^2 \left[\int_{\eta_1}^{\eta_3} \frac{f_3(\eta)}{(\eta-y)^2 + z^2}\, d\eta \right] dy \qquad (24)$$

It is shown in Appendix D that $\lim_{z\to 0} \frac{\partial}{\partial z}$ (...) applied to the second term in the bracket of I_{313} is zero. The first term is independent of z. Thus,

$$\lim_{z\to 0} \frac{\partial}{\partial z} (zJ_{313}) = \int_{y_1}^{y_3} f_1(y)\left(\int_{\eta_1}^{\eta_3} f_3(\eta,y)d\eta\right)dy$$

The integrands are analytic functions.

The inner integration in I_{311} is carried out analytically.

$$\int_{\eta_1}^{\eta_3} \frac{d\eta}{(\eta-y)^2 + z^2} = \frac{1}{z} \operatorname{arctg} \frac{\eta-y}{z} \Bigg|_{\eta_1}^{\eta_3} \tag{25}$$

One then has to evaluate

$$\lim_{z\to 0} \frac{\partial}{\partial z} z \frac{1}{z}[\operatorname{arctg} \frac{\eta-y}{z}] \Bigg|_{\eta_1}^{\eta_3} = \lim_{z\to 0} - \frac{\eta-y}{(\eta-y)^2 + z^2} \Bigg|_{\eta_1}^{\eta_3}$$

$$= - \frac{1}{\eta_3-y} + \frac{1}{\eta_1-y} \tag{26}$$

This result could have been obtained by setting z = 0 on the left side of Eq. (25), and substituting into the formal expression for the indefinite integral (namely $-\eta^{-1}$) the limits η_1 and η_3; this, in spite of the fact that the integral obtained by immediately setting z = 0 does not exist for η = 0. Then

$$\lim_{z\to 0} \frac{\partial}{\partial z} (z\ I_{311}) = \beta^2 \int_{y_1}^{y_3} f_1(y)f_2(y)\left(- \frac{1}{\eta_3-y} + \frac{1}{\eta_1-y}\right)dy$$

The integrand is singular for $y = \eta_3$ and $y = \eta_1$. If the xy-element lies exactly in the wake of the $\xi\eta$-element, then $\eta_1 = y_1$, $\eta_3 = y_3$; the integrand is singular at the two limits. If the xy-element is adjacent to the wake, then this happens at only one limit. The two singular terms are now treated separately.

$$- \int_{y_1}^{y_3} \frac{f_1(y) f_2(y)}{\eta_3 - y} \, dy = - f_1(\eta_3) f_2(\eta_3) \int_{y_1}^{y_3} \frac{dy}{\eta_3 - y}$$

$$- \int_{y_1}^{y_3} \left[\frac{f_1(y) f_2(y) - f_1(\eta_3) f_2(\eta_3)}{\eta_3 - y} \right] dy \tag{26a}$$

The second term on the right is a smooth function. It, therefore, can be integrated numerically without difficulty. It is best to write it in the form

$$- \int_{y_1}^{y_3} \left[\frac{f_1(y) - f_1(\eta_3)}{\eta_3 - y} f_2(y) + \frac{f_2(y) - f_2(\eta_3)}{\eta_3 - y} f_1(\eta_3) \right] dy$$

The first term on the right of Eq. (26a) gives

$$f_1(\eta_3) f_2(\eta_3) \log \left| \frac{\eta_3 - y_3}{\eta_3 - y_1} \right|$$

This term is infinite if $y_3 = \eta_3$. This happens because at the element boundaries the upwash caused by the $\xi\eta$-element behaves as $(\eta_3 - y)^{-1}$. If $\Delta\bar{p}$ is continuous as one passes over the element boundaries in the η-direction (as it happens for the two boundaries between the smaller triangles in Fig. 9 or for the triangular elements with one side parallel to the ξ-axis and linear pressure distributions), then the singular term is canceled by a contribution of the adjacent $\xi\eta$-element.

We have

$$\lim_{z\to 0} \frac{\partial}{\partial z} (z\ I_{311})$$

$$= \beta^2 \left\{ f_1(\eta_3) f_2(\eta_3) \log \frac{\eta_3 - y_3}{\eta_3 - y_1} - f_1(\eta_1) f_2(\eta_1) \log \frac{\eta_1 - y_3}{\eta_1 - y_1} \right.$$

$$\left. - \int_{y_1}^{y_3} \frac{f_1(y)f_2(y) - f_1(\eta_3)f_2(\eta_3)}{\eta_3 - y}\, dy + \int_{y_1}^{y_3} \frac{f_1(y)f_2(y) - f_1(\eta_1)f_2(\eta_1)}{\eta_1 - y} \right\} dy \tag{27}$$

The inner integral of I_{312} gives

$$\frac{1}{2} \log((\eta - y)^2 + z^2) \Big|_{\eta_1}^{\eta_3}$$

One has

$$(1/2) \lim_{z\to 0} \frac{\partial}{\partial z} (z \log((\eta - y)^2 + z^2)) \Big|_{\eta_1}^{\eta_3}$$

$$= \lim_{z\to 0} \left[(1/2)\log((\eta - y)^2 + z^2) \Big|_{\eta_1}^{\eta_3} + \frac{z^2}{(\eta - y)^2 + z^2} \Big|_{\eta_1}^{\eta_3} \right] = \log|\eta - y| \Big|_{\eta_1}^{\eta_3}$$

This result would have been obtained, if one sets in Eq. (23) $z = 0$, disregards the singularity of the integrand, integrates analytically, and substitutes the limits.

Then

$$\lim_{z\to 0} \frac{\partial}{\partial z}(zJ_{312}) = \beta^2 \int_{y_1}^{y_3} f_1(y)f_2'(y)[\log(\eta_3-y) - \log(\eta_1-y)]dy \qquad (28)$$

The integrand is singular for $y = \eta_3$ and $y = \eta_1$. We introduce

$$f_4(y) = \int_{y_0}^{y} f_1(v)f_2'(v)dv \qquad (29)$$

The first term of Eq. (28) can then be written (for q fixed)

$$\int_{y_1}^{y_3} \frac{d}{d\eta} [f_4(y) - f_4(q)]\log(\eta_3-y)dy$$

$$= (f_4(y) - f_4(q))\log(\eta_3-y) \Big|_{y_1}^{y_3} + \int_{y_1}^{y_3} \frac{(f_4(y) - f_4(q))}{\eta_3-y} dy \qquad (30)$$

If η_3 lies within the interval of integration, one chooses $q = \eta_3$. This ensures that $(f_3(y) - f_3(q)/(\eta_3-y)$ is an analytic function of y, even at $y = \eta_3$. If η_3 lies outside of this region, then the integrand is analytic in any case. It is, however, still advisable to choose the constant $f_4(q)$ so that the integrand is as smooth as possible, but it is not necessary that one evaluate Eq. (29) for values of y outside of $y_1 < y < y_3$, or if one does, it need not be done with precision. Therefore,

$$I_{312} = \beta^2(f_4(y) - f_4(\eta_3))\log(\eta_3 - y) \Big|_{y_1}^{y_3} + \int_{y_1}^{y_3} \frac{f_4(y) - f_4(\eta_3)}{\eta_3 - y}\, d\eta$$

$$- (f_4(y) - f_4(\eta_1)\log(\eta_1 - y) \Big|_{y_1}^{y_3} - \int_{y_1}^{y_3} \frac{f_4(y) - f_4(\eta_1)}{\eta_1 - y}\, d\eta \tag{31}$$

Let I_{32} be the integral I with K replaced by K_{32}, Eq. (14). We omit a demonstration which would show that

$$\lim_{z\to 0}\left(\frac{\partial}{\partial z}(zI_{32}\right) = \lim_{z\to 0} I_{32}$$

Let

$$f_5(\eta - y) = \sum_{n=0}^{\infty} \frac{(k(\eta-y)/2\beta)^{2n}}{n!(n+1)!} \tag{32}$$

Then

$$\lim_{z\to 0} I_{32} = k^2 \int_{y_1}^{y_3} f_1(y)\left(\int_{\eta_1}^{\eta_3} \log(\eta - y) f_5(\eta - y) f_2(\eta)\, d\eta\right) dy \tag{33}$$

Let

$$f_6(\eta, y) = \int_{v=y}^{v=\eta} f_2(v) f_5(v-y)\, dv \tag{34}$$

The inner integral then becomes

$$\int_{\eta_1}^{\eta_3} \log|\eta - y|(\partial f_6/\partial\eta)\, d\eta = (\log|\eta - y| f_6(\eta, y) \Big|_{\eta=\eta_1}^{\eta=\eta_3} - f_7(y) \tag{35}$$

with

$$f_7(y) = \int_{\eta_1}^{\eta_3} \frac{f_6(\eta,y)}{\eta-y}\, d\eta \tag{36}$$

The integrand in the last expression is a smooth function of η which can be treated numerically.

For the evaluation of I_{32}, Eq. (33), one then needs

$$-(k^2/2) \int_{y_1}^{y_3} f_1(y) f_7(y) dy \tag{37}$$

and

$$k^2/2 \int_{y_1}^{y_3} f_1(y)[\log|\eta_3-y|f_6(\eta_3,y) - \log|\eta_1-y|f_6(\eta_1,y)]dy \tag{38}$$

The integrand of Eq. (37) is smooth. The two terms in the bracket of the integrand in Eq. (38) are treated separately. Let

$$f_8(y,\eta_3) = \int_{y_0}^{y} f_1(v) f_6(\eta_3,v) dv \tag{39}$$

Then one obtains instead of the first of these two terms

$$\begin{aligned} &\int_{y_1}^{y_3} \frac{\partial}{\partial z} [f_8(y,\eta_3) - f_8(q,\eta_3)]\log|\eta_3-y|dy \\ &= \log|\eta_3-y|[f_8(y,\eta_3-f_8(q,\eta_3)\Big|_{y_1}^{y_3} - f_9(\eta_3) \end{aligned} \tag{40}$$

where

$$f_9(\eta_3) = \int_{y_1}^{y_3} \frac{f_8(\eta,\eta_3) - f_8(q,\eta_3)}{y-\eta_3} dy \qquad (41)$$

If $y_1 < \eta_3 < y_3$, one will choose $q = \eta_3$. Then the integrand of Eq. (41) is an analytic function, even at $y = \eta_3$. If η_3 is outside of this region then the integrand of Eq. (41) is analytic for any choice of q in the constant $f_7(q,\eta_3)$, but one will choose q approximately equal to η_3 to obtain a smooth integrand of Eq. (41) in the interval of integration. The same procedure is applied to the other term, with η_3 replaced by η_1.

Then

$$I_{32} = (k^2/2) - \left\{ \int_{y_1}^{y_3} f_1(y) f_7(y) dy \right.$$

$$+ \log(\eta_3 - y)[f_8(y,\eta_3) - f_8(q,\eta_3)] \Big|_{y_1}^{y_3} - f_9(\eta_3) \qquad (42)$$

$$\left. - \log(\eta_1 - y)[f_8(y,\eta_1) - f_8(q,\eta_1)] \Big|_{y_1}^{y_3} + f_9(\eta_1) \right\}$$

For triangular elements the functions f_1 and f_2 (Eqs. (19)) are given by

$$f_1(y,y_1,c_1) = c_1(y-y_1)$$

$$f_2(\eta,\eta_1,c_1,c_3) = c_2(\eta-\eta_1) + c_3(\eta-\eta_1)^2$$

The constants c_1, c_2, and c_3 are the only parameters by which the elemental pressure distributions enter the computations. It is therefore practical to program their contributions to the following expressions separately then

$$f_3 = c_3$$

and

$$I_{313} = c_1 c_3 (y_3 - y_1)^2/2$$

One has

$$(f_1(y)f_2(y) - f_1(v)f_2(v))(v-y)^{-1}$$

$$= [(f_1(y) - f_1(v))/(v-y)]f_2(y) + [(f_2(y) - f_2(v))/(v-y)]f_1(v)$$

$$= -c_1\{c_2(y-\eta_1) + c_3(y-\eta_1)^2 + (v-y_1)[c_2 + c_3(y-\eta_1 + v-\eta_1]\}$$

Then

$$\int(f_1(y)f_2(y) - f_1(v)f_2(v)(v-y)^{-1}dy$$

$$= -c_1c_2[(v-y)_1(y-\eta_1) + (y-\eta_1)^2/2]$$

$$-c_1c_3[(v-y_1(v-\eta_1)(y-\eta_1) - ((v-y_1)(y-\eta_1)^2/2) + (y-\eta_1)^3/3]$$

In the program this is considered as a function of y, v, y_1, η_1, (c_1c_2), and (c_1c_3). The expression occurring in Eq. (26a) are then obtained by substituting y_3 or y_1 and η_3 or η_1 for y and v respectively. The lower limit in the integral for f_4 does not matter.

The integrand in f_4 is given by

$$c_1(v-y_1)[c_2 + 2c_3\ (v-y_1 + y_1 - \eta_1)]$$

Then

$$f_4(y) = c_1c_2[(y-y_1)^2/2) + 2c_1c_3[((y_1-\eta_1)(y-y_1)^2/2) + ((y-y_1)^3/3)]$$

programmed as a function of y, y_1, η_1, (c_1c_2), and (c_1c_3).

$$\int(f_4(y) - f_4(v))(v-y)^{-1}dy = -(c_1c_2/2)[(v-y_1)(y-y_1) + (y-y_1)^2/2)]$$

$$-c_1c_3[(y_1-\eta_1)((v-y_1)(y-y_1) + (y-y_1)^2/2) + (2/3)(v-\eta_1)^2(y-y_1)$$

$$+ (1/3)(v-y_1)(y-y_1)^2 + (2/9)(y-y_1)^3]$$

programmed as a function of y, v, y_1, η_1, (c_1c_2), and (c_1c_3). The integral on the right of Eq. (30) is then obtained by substituting y_3 and y_1 for y, and η_3 for v. The same procedure is carried out for the second term in the integral on the right of Eq. (28); one simply replaces η_3 by η_1.

The terms of I_{32} are $O(k^2)$, moreover η-y is small because these computations will be carried out only if the xy-elements are in the vicinity of the wake of the $\xi\eta$-element. Therefore, an approximation of f_5 by its first term will suffice

$$f_5 = 1$$

Then

$$f_6(\eta,y) = \int_{v=y}^{v=\eta} f_2(v)dv$$

$$= ((c_2/2)(\eta-\eta_1)^2 - (y-\eta_1)^2) + ((c_3/3)(\eta-\eta_1)^3 - (y-\eta_1)^3)$$

For the evaluation of f_8 this expression is reordered in powers of $(y-y_1)$

$$f_6(\eta,y) = (c_2/2)[(\eta-\eta_1)^2 - (y_1-\eta_1)^2 - 2(y_1-\eta_1)(y-y_1) - (y-y_1)^2]$$
$$+ (c_3/3)[(\eta-\eta_1)^3 - (y_1-\eta_1)^3 - 3(y_1-\eta_1)^2(y-y_1)$$
$$- 3(y_1-\eta_1)(y-y_1)^2 - (y-y_1)^3]$$

For the evaluation of f_7 the first form of f_6 is used

$$f_7(y) = \int_{\eta_1}^{\eta_3} f_6(\eta,y)(\eta-y)^{-1}dy$$

$$f_7(y) = (c_2/2)[((\eta_3-\eta_1)^2/2) + (y-\eta_1)(\eta_3-\eta_1)]$$
$$+ (c_3/3)[((\eta_3-\eta_1)^3/3) + ((y-\eta_1)(\eta_3-\eta_1)^2/2) + (y-\eta_1)^2(\eta_3-\eta_1)]$$

This is reorderd, in powers of $(y-y_1)$

$$f_7(y) = (c_2/2)[((\eta_3-\eta_1)^2/2) + (y_1-\eta_1)(\eta_3-\eta_1) + (\eta_3-\eta_1)(y-y_1)]$$
$$+ (c_3/3)\{[((\eta_3-\eta_1)^3/3) + ((y_1-\eta_1)(\eta_3-\eta_1)^2/2) + (y_1-\eta_1)^2(\eta_3-\eta_1)]$$
$$+ [((\eta_3-\eta_1)^2/2)+ 2(y_1-\eta_1)(\eta_3-\eta_1)](y-y_1) + (\eta_3-\eta_1)(y-y_1)^2\}$$

Then

$$\int_{y_1}^{y_3} f_1(y)f_7(y)dy = (c_1c_2/2)\{[((\eta_3-\eta_1)^2/2)$$
$$+ (y_1-\eta_1)(\eta_3-\eta_1)](y_3-y_1)^2/2 + (\eta_3-\eta_1)(y_3-y_1)^3/3\}$$
$$+ (c_1c_3/3)\{[((\eta_3-\eta_1)^3/3) + ((y-\eta_1)(\eta_3-\eta_1)^2/2)$$
$$+ (\eta_1-y_1)^2(\eta_3-\eta_1)](y_3-y_1^2)/2 + [((\eta_3-\eta_1)^2/2)$$
$$+ 2(y_1-\eta_1)(\eta_3-\eta_1)]((y_3-y_1)^3/3) + (\eta_3-\eta_1)((y-y_1)^4/4)\}$$

$$f_8(y,\eta) = \int^{y} f_1(v) f_6(\eta,v)\,dv$$
$$= (c_1c_2/2)\{[(\eta-\eta_1)^2 - (y_1-\eta_1)^2]((y-y_1)^2/2) - 2(y_1-\eta_1)((y-y_1)^3/3)$$
$$- (y-y_1)^4/4]\} + (c_1c_3/3)\{[(\eta-\eta_1)^3 - (y_1-\eta_1)^3]((y-y_1)^2/2)$$
$$- 3(y_1-\eta_1)^2((y-y_1)^3/3) - 3(y_1-\eta_1)((y-y_1)^4/4) - ((y-y_1)^5/5)\}$$

This is programmed as a function with arguments $y-y_1$, $\eta-\eta_1$, $y_1-\eta_1$, (c_1c_2), and (c_1c_3).

Finally, one needs

$$f_9(\eta,q) = \int_{y_1}^{y_3} \frac{f_8(y,\eta) - f_8(q,\eta)}{y-q}\,dy$$

but only for $\eta = q$. Ultimately, one must substitute $q = \eta_3$ and $q = \eta_1$.

$$f_9(q,q) = (c_1c_2/2)\{[(q-y_1)^2 - (y_1-\eta_1)^2][(y_3-y_1)^2/4$$
$$+ ((q-y_1)(y_3-y_1)/2)] - 2(y_1-\eta_1)[((y_3-y_1)^3/9) + (q-y_1)((y_3-y_1)^2/6)$$
$$+ (q-y_1{}^2)((y_3-y_1)/3)] - [((y_3-y_1)^4/16) + (q-y_1)((y_3-y_1)^3/12)$$
$$+ (q-y_1)^2((y-y_1)^2/8) + (q-y_1)^3((y_3-y_1)/4)]\}$$
$$+ (c_1c_3/3)\{[(q-\eta_1)^3 - (y_1-\eta_1)^3][((y_3-y_1)^2/4) + (q-y_1)((y_3-y_1)/2))]$$
$$- 3(y_1-\eta_1)^2[((y_3-y_1)^3/9) + (q-y_1)((y_3-y_1)^2/6) + (q-y_1)^2((y_3-y_1)/3)]$$
$$- 3(y_1-\eta_1)[((y_3-y_1)^4/16) + (q-y_1)((y_3-y_1)^3/12) + (q-y_1)^2((y_3-y_1)^2/8)$$
$$+ (q-y_1)^3((y_3-y_1)/4)] - [((y_3-y_1)^5/25) + (q-y_1)((y_3-y_1)^4/20)$$
$$+ (q-y_1)^2((y_3-y_1)^3/15) + (q-y_1)^3((y_3-y_1)^2/10) + (q-y_1)^4((y_3-y_1)/5)]\}$$

SECTION IV

THE LIMITING PROCESS $z \to 0$ FOR POINTS (x,y) CLOSE TO OR WITHIN A $\xi\eta$-ELEMENT

A more refined procedure is needed for points (x,y) close to points (ξ,η) because certain terms in K have a denominator R, which tends to zero if $(\xi-x)$ and $(\eta-y)$ tend to zero simultaneously, and because for $\xi-x$ negative there are terms with a denominator $r = ((\eta-y)^2 + z^2)^{1/2}$. The latter singularity has been treated in a simpler situation in Section III. In the present section we study the limit $z \to 0$ for the upwash at a fixed point (x,y).

According to Eqs. (B.64) one must evaluate

$$\bar{w}(x,y,z=0) = \lim_{z\to 0} \frac{\partial}{\partial z} \left\{ z \iint_{A(\xi,\eta)} \Delta\bar{p}(\xi,\eta)(K_1 + K_2)d\xi d\eta \right\} \qquad (43)$$

where

$$\bar{w}(x,y,z) = 8\pi\beta \exp(ikx) w(x,y,z)$$

The function $K = K_1 + K_2$ depends only upon $\xi-x$ and r with

$$r = ((\eta-y)^2 + z^2)^{1/2}$$

The independent variable z occurs only in r. Because of the singularities in K and because of the differentiation with respect to z the result of the limiting process $z \to 0$ is not entirely self-evident. For part of the discussion we introduce polar coordinates

$$\xi - x = \rho\cos\alpha, \quad \eta - y = \rho\sin\alpha$$

Then

$$R^2 = \rho^2 + z^2 \qquad (44)$$

$$d\xi d\eta = \rho d\rho d\alpha$$

One deals with integrals

$$\iint F(\rho,\alpha)d\rho d\alpha$$

Expressing $F(\rho,\alpha)$ in the form

$$F(\rho,\alpha) = \partial G(\rho,\alpha)/\partial\alpha \qquad (45)$$

one obtains by an argument familiar from the derivation of Green's theorem

$$\iint_A F(\rho,\alpha)d\rho d\alpha = \iint_A (\partial G(\rho,\alpha)/\partial\alpha)d\rho d\alpha = -\oint G(\rho,\alpha)d\rho \qquad (46)$$

Alternatively one can set

$$F(\rho,\alpha) = \partial H(\rho,\alpha)/\partial\rho \qquad (47)$$

Then one obtains

$$\iint_A F(\rho,\alpha)d\rho d\alpha = \iint_A(\partial H(\rho,\alpha)/\partial\rho)d\rho d\alpha = \oint H(\rho,\alpha)d\alpha \qquad (48)$$

If the point xy lies outside the $\xi\eta$-element, then $\rho \neq 0$ throughout the element, and α returns after one complete circuit around the element to its original value (Fig. 11). If it lies inside, then ρ varies between 0 and $\rho(\alpha)$, and α between 0 and 2π (Fig. 12).

For part of the discussion it is convenient to write

$$H(\rho,\alpha) = \int_0^\rho F(\tilde{\rho},\alpha)d\tilde{\rho}$$

(where $\tilde{\rho}$ is a dummy variable of integration). Then

$$\iint_A F(\rho,\alpha)d\rho d\alpha = \int_{\ell=0}^{L} \left(\int_0^{\rho(\alpha)} F(\tilde{\rho},\alpha)d\rho \right)(d\alpha/d\ell))d\ell$$

Here ℓ is the length measured along the contour of the element (or some other parameter which varies monotomically from 0 to L). The value of α along the contour is considered as function of ℓ.

We start the discussion of the individual terms of K with K_2, Eq. (B.65). Only the first term, $-R^{-1}[(\exp(ikV) - 1)/V]$, has a denominator which tends to zero as ρ and z tend to zero, namely R^{-1}. The expression V has the same property, but $(\exp(ikV) - 1)/V$ is obviously regular at V = 0. If any term is critical in the limiting process $z \to 0$, then it is the first one namely $-(ik)/R$. It is discussed presently. Except for factors one deals with the expression

$$\iint \Delta\bar{p}(\rho,\alpha)R^{-1}\rho d\rho d\alpha = \int_0^L \int_{\rho=0}^{\rho(\alpha)} (\Delta\bar{p}(\rho,\alpha)(\rho^2 + z^2)^{-1/2}\rho d\rho)(d\alpha/d\ell)\ell \quad (49)$$

For fixed α the inner integral has the form

$$I = \int_0^{\rho(\alpha)} f(\rho)(\rho^2 + z^2)^{-1/2}\rho d\rho$$

Ultimately one has to form $\lim_{z \to 0} (z \frac{\partial}{\partial z} \ldots)$ for this expression. By an integration by parts one obtains

$$I = f(\rho)(\rho^2 + z^2)^{1/2} \Big|_0^{\rho(\alpha)} - \int_0^{\rho(\alpha)} (df/d\rho)(\rho^2 + z^2)^{1/2}d\rho$$

and

$$\frac{\partial}{\partial z}(zI) = f(\rho)(\rho^2 + z^2)^{1/2}\Big|_0^{\rho(\alpha)} - \int_0^{\rho(\alpha)} (df/d\rho)(\rho^2 + z^2)^{1/2} d\rho$$

$$+ z^2\Big[f(\rho)(\rho^2 + z^2)^{-1/2}\Big|_0^{\rho(\alpha)} - \int_0^{\rho(\alpha)} (df/d\rho)(\rho^2 + z^2)^{-1/2} d\rho\Big]$$

The limiting process $z \to 0$ gives

$$\lim_{z\to 0}\Big(\frac{\partial}{\partial z}(zI)\Big) = f(\rho(\alpha))\rho(\alpha)) - \int_0^{\rho(\alpha)} (df/d\rho)\rho d\rho$$

$$+ \lim_{z\to 0}\Big[z^2 f(\rho(\alpha))\rho(\alpha)^{-1} - zf(0) - z^2 \int_0^{\rho(\alpha)} (df/d\rho)(\rho^2 + z^2)^{-1/2} d\rho\Big]$$

In the first line one carries out an integration by part and obtains

$$\int_0^{\rho(\alpha)} f(\rho) d\rho$$

The first two terms of the second line vanish for $z = 0$. In the last term one has

$$z^2 \left| \int_0^{\rho(\alpha)} (df/d\rho)(\rho^2 + z^2)^{-1/2} d\rho \right| < z^2 \max|df/d\rho| \int_0^{\rho\alpha} (\rho^2 + z^2)^{-1/2} d\rho$$

Now

$$\int_0^{\rho(\alpha)} (\rho^2 + z^2)^{-1/2} d\rho = \log(\rho + (\rho^2 + z^2)^{1/2})\Big|_0^{\rho(\alpha)}$$

The lower limit might be critical; it gives log z. The entire expression is, however, $O(z^2 \log z)$.

One thus obtains

$$\lim_{z \to 0} \frac{\partial}{\partial z}\left(z \int_0^{\rho(\alpha)} f(\rho)(\rho^2 + z^2)^{-1/2}\rho d\rho\right) = \int_0^{\rho(\alpha)} f(\rho)d\rho$$

Applying this result to Eq. (49) one has

$$\iint \Delta\bar{p}(\rho,\alpha)R^{-1}\rho d\rho d\alpha = \int_0^L \left(\int_{\rho=0}^{\rho(\alpha)} (\Delta\bar{p}(\rho,\alpha)d\rho\right)(d\alpha/d\ell)d\ell \qquad (50)$$

The results for K_1 are less foreseeable. As $\rho \to 0$, the first term of K_1 behaves for $z = 0$ as ρ^{-2}, moreover, there is a singular point in the wake of the point (ξ,η), even for $\rho \neq 0$.

For a fixed point (x,y) the function $\Delta\bar{p}(\xi,\eta)$ is written in the form

$$\Delta\bar{p}(\xi,\eta) = \sum_m \sum_n c_{mn}(x,y)(\xi-x)^m(\eta-y)^n \qquad (51)$$

obviously

$$c_{00} = \Delta\bar{p}(x,y)$$

In practice $m \leq 1$, $n \leq 1$, but the analysis will be carried out in more general terms. The individual terms in Eq. (51) will be treated separately.

The term $\beta^2[R(R + \xi-x)]^{-1}$ of K_1 gives a contribution to the upwash of the form

$$\beta^2 \int (\xi-x)^m(\eta-y)^n[R(R + \xi-x)]^{-1}d\xi d\eta \qquad (52)$$

To this expression we apply the procedure implied by Eq. (45), (rather than Eq. (47) because the integration with respect to α is somewhat more easily discussed than the integration with respect to ρ). Setting

$$\phi = \alpha - \pi \tag{53}$$

(because then an important singularity occurs at $\phi = 0$), one obtains for a single term of the expression in Eq. (51)

$$\begin{aligned} I_{mn} = (-)^{m+n}\beta^2 \iint_A \rho^{m+n-1} \cos\phi^m \sin\phi^n \\ \left[\left(1 + \left(\frac{z^2}{\rho^2}\right)\right)^{1/2} \left(\left(1 + \frac{z^2}{\rho^2}\right)^{1/2} - \cos\phi\right)\right]^{-1} d\rho d\phi \end{aligned} \tag{54}$$

The function G of Eq. (45) is then defined by

$$G(\rho,\phi) = (-)^{m+n}\beta^2\rho^{m+n-1}a^{-1}g(a,\phi) \tag{55}$$

with

$$g(a,\phi) = \int \cos^m\phi \sin^n\phi [a - \cos\alpha]^{-1} d\phi \tag{56}$$

and

$$a(z,\rho) = (1 + (z^2/\rho^2)^{1/2} \tag{57}$$

One has

$$\begin{aligned} a^2 - 1 &= z^2/\rho^2 \\ \partial a/\partial z &= a^{-1}\rho^{-2}z \end{aligned} \tag{58}$$

If the point xy lies outside the $\xi\eta$-element, then $\rho > 0$ for all points of the $\xi\eta$-element and the denominator ρ^{-2} in Eq. (57) is no matter of concern. If the point xy lies inside the $\xi\eta$-element, then one introduces a cut in the $\xi\eta$-plane in the downstream direction through the point xy. In the $\rho\phi$ plane one then obtains the region shown in Fig. 13. The boundaries AB and DE are respectively lines $\phi = -\pi$ and $\phi = +\pi$. Along the contour of the element, ρ is considered as a function of ϕ, denoted by $\rho(\phi)$. Along AB and DE, ρ varies from zero to $\rho(-\pi) = \rho(\pi)$.

The expression in Eq. (54) is now evaluated by means of Eq. (46). One obtains

$$\tilde{w} = \beta^2(-)^{m+n+1} \lim_{z\to 0}\left(\frac{\partial}{\partial z} z \int \rho^{m+n-1} a^{-1} g(a,\phi) d\rho\right) \tag{59}$$

Using the definitions in Eqs. (55) through (57) one has a combined contribution of AB and DE (Fig. 13).

$$(-)^{m+n+1}\beta^2 \int_0^{\rho(\pi)} \rho^{m+n-1} a^{-1}(g(a,\pi) - g(a,-\pi))d\rho$$

For <u>n odd</u> the integrand in Eq. (56) is an odd function of ϕ, therefore,

$$g(a,\pi) - g(a,-\pi) = 0 \quad , \quad n \text{ odd}$$

and one only needs to consider the integral along BCD, which amounts to the integral around the contour of the region in the $\xi\eta$-plane.

$$I_{mn} = (-)^{m+n+1}\beta^2 \oint \rho^{m+n-1} a^{-1} g(a,\phi) d\rho \tag{60}$$

The function $g(a,\phi)$ is defined in Eq. (56). The variables z and ρ enter through the parameter "a," Eq. (57). The integral is formed along the contour of the region.

To evaluate $g(a,\phi)$ (Eq. (56)) we set, for n odd

$$g(a,\phi) = g_1(a,\phi) + g_2(a,\phi) \tag{61}$$

with

$$g_1(a,\phi) = -\int[\cos^m\phi\sin^{n-1}\phi - a^m(1 - a^2)^{(n-1)/2}][\cos\phi - a]^{-1}\sin\phi d\phi$$

$$g_2(a,\phi) = \int a^m(1 - a^2)^{(n-1)/2}[a - \cos\phi]^{-1}\sin\phi d\phi \tag{62}$$

In the last equation, Eq. (58) is substituted and the integration is carried out.

$$g_2 = (-)^{(n-1)/2}a^m(z/\rho)^{n-1}\log(a - \cos\phi) \tag{63}$$

In g_1 one can carry out the division by $(\cos\phi - a)$. One obtains a polynomial in $\cos\phi$ with coefficient given by powers of "a." Notice that

$$\partial g_1(1,\phi)/\partial\phi = \cos^m\phi\sin^n\phi(1 - \cos\phi)^{-1} \quad \text{for } n \geq 3, \text{ n odd}$$

$$\partial g_1(1,\phi)/\partial\phi = (\cos^m\phi - 1)(1 - \cos\phi)^{-1}\sin\phi \quad \text{for } n = 1 \tag{64}$$

Let

$$I_{mn} = I_{mn,1} + I_{mn,2} \tag{65}$$

with

$$I_{mn,i} = (-)^{m+n+1}\beta^2 \oint \rho^{m+n-1}a^{-1}g_i d\rho, \quad i = 1,2$$

To obtain the upwash $\bar{w}$, one must form $\lim_{z\to 0} \frac{\partial}{\partial z}(zI_{mn})$. Along the contour $\rho \neq 0$. In the present case (n odd) one obtains the result immediately by setting $a = 1$.

$$\bar{w}_{mn,1} = (-)^{m+n+1} \beta^2 \int_{\ell=0}^{L} \rho^{m+n-1}g_1(1,\phi)(d\rho/d\ell)d\ell$$

where $d\ell$ is the length element of the contour and L the total length. Carrying out an integration by parts and using Eq. (64) one obtains

$$\bar{w}_{mn,1} = (-)^{m+n}\beta^2(m+n)^{-1}\int_{\ell=0}^{L} \rho^{m+n}\cos^m\phi\sin^n\phi(1 - \cos\phi)^{-1}(d\phi/d\ell)d\ell \tag{66}$$

$$n > 3,\ n \text{ odd}$$

$$\bar{w}_{m1,1} = (-)^{m+1}\beta^2(m+1)^{-1}\int_{\ell=0}^{L} \rho^{m+1}(\cos^m\phi - 1)(1 - \cos\phi)^{-1}\sin\phi(d\phi/d\ell)d\ell \tag{67}$$

$$n = 1$$

With Eq. (63) one obtains

$$I_{mn,2} = \beta^2(-)^{m+n+1+(n-1)/2}z^{n-1}\int \rho^m a^{m-1}\log(a - \cos\phi)d\rho$$

Because of the factor z^{n-1}, $I_{mn,2}$ will be zero for $n \geq 3$. The discussion is therefore restricted to the case $n = 1$. There one obtains

$$\begin{aligned}
\bar{w}_{m,1,2} &= \lim_{z\to 0}\frac{\partial}{\partial z}(zI_{m1,2}) \\
&= \lim_{z\to 0}(-)^m\Bigg\{\int_0^L \rho^m\ a^{m-1}\log(a - \cos\phi)(d\rho/d\ell)d\ell \\
&\quad + z^2(m-1)\int_0^L \rho^{m-2}\ a^{m-3}\log(a - \cos\phi)(d\rho/d\ell)d\ell \\
&\quad + z^2\int_0^L \rho^{m-2}\ a^{m-1}[a - \cos\phi]^{-1}(d\rho/d\ell)d\ell\Bigg\}
\end{aligned} \tag{68}$$

If $\phi \neq 0$ along the entire contour, one obtains immediately

$$\bar{w}_{m,1,2} = (-)^m \oint \rho^m \log(1 - \cos\phi) d\rho$$

and after an integration by parts

$$\bar{w}_{m,1,2} = (-)^{m+1} \oint (m+1)^{-1} \rho^{m+1}(1 - \cos\phi)^{-1} \sin\phi d\phi$$

$$\text{for } n = 1$$

We observed above that $\bar{w}_{m,n,2} = 0$ for $n \geq 3$. One then obtains from Eq. (66) and (67)

$$\bar{w}_{mn} = (-)^{m+n} \beta^2 (m+n)^{-1} \oint \rho^{m+n} \cos^m\phi \sin^n\phi (1 - \cos\phi)^{-1} d\phi$$

We return to $\alpha = \phi + \pi$, and subsequently to the original coordinates $(\xi-x)$ and $(\eta-y)$. One has

$$d\alpha = [(\xi-x)d\eta - (\eta-y)d\xi]/\rho^2$$

Therefore,

$$\bar{w}_{mn} = \beta^2 (m+n)^{-1} \int (\xi-x)^m (\eta-y)^n [\rho + (\xi-x)]^{-1} \rho^{-1} [(\xi-x)d\eta - (\eta-y)d\xi] \tag{69}$$

The result remains the same if at some point of the contour $\phi = 0$. There $\eta - y = 0$, and $\xi-x < 0$. Then one writes

$$(\rho + (\xi-x))^{-1} = (\rho - (\xi-x)/(\rho^2 - (\xi-x)^2)$$

$$= \frac{\rho - (\xi-x)}{(\eta-y)^2}$$

One thus obtains an integrand

$$(\xi-x)^m(\eta-y)^{n-2}(\rho - (\xi-x))\rho^{-1}[(\xi-x)d\eta - (\eta-y)d\xi]$$

If $n > 3$, then there is no singularity. If $n = 1$, there is one singular point where the integrand behaves as $(\eta-y)^{-1}$. The value of $\bar{w}_{m1}$ is then obtained by taking the principal value.

This is seen from Eq. (67). Let ℓ_0 be the value of ℓ for which $\phi = 0$. In the vicinity of this point we consider ℓ as variable of integration. Let $\ell = \ell(\phi)$. Then $\ell(-\pi) = 0$, $\ell(\pi) = L$, and $\ell(0) = \ell_0$. In the first integral in Eq. (68), but with limits

$$\int_0^{\ell(-\varepsilon)} + \int_{\ell(+\varepsilon)}^{L},$$

one can make the limiting process $z \to 0$ and obtains

$$(-)^n \int_0^{\ell(-\varepsilon)} + \int_{\ell(+\varepsilon)}^{L} \rho^m \log(1 - \cos\phi)(d\rho/d\ell)d\ell$$

In the region $-\varepsilon < \phi < \varepsilon$, we take ϕ as variable of integration

$$\int_{-\varepsilon}^{+\varepsilon} \rho^m a^{m-1}(d\rho/d\phi)\log(a^2 - \cos^2\phi) - \log(a + \cos\phi)]d\phi$$

This is majorized by

$$\int_{-\varepsilon}^{\varepsilon} \text{const} \log (a^2 - \cos^2\phi)\cos\phi d\phi + \int \text{const } d\phi$$

$$= \text{const} \int_{-\varepsilon}^{+\varepsilon} \log(\frac{\rho^2}{z^2} + \sin^2\phi)d(\sin\phi) + \text{const } 2\varepsilon$$

and, by an integration by part

$$= \text{const}\left[\sin\phi\log\left(\frac{\rho^2}{z^2} + \sin^2\phi\right)\Bigg|_{-\varepsilon}^{+\varepsilon} - \int_{-\varepsilon}^{+\varepsilon}\frac{2\sin\phi\, d(\sin\phi)}{\frac{z^2}{\rho^2} + \sin^2\phi}\right] + \text{const } 2\varepsilon$$

The integral is $O(\varepsilon)$, the first term $O(\varepsilon\log\varepsilon)$. As expected the term vanishes in the limit $\varepsilon \to 0$. The first one can be replaced by

$$\displaystyle{\not}\!\!\int_0^L \rho^m \log(1 - \cos\phi)(d\rho/d\ell)d\ell$$

where the slash indicates that an ε neighborhood of $\ell = \ell_0$ is omitted. Actually, in this form the integral is well defined even if one does not omit this neighborhood. But if one carries out an integration by parts, as one does in order to arrive at Eq. (69), the exclusion of such a neighborhood is necessary and leads to the definition of the principal part of the integral. The second integral in Eq. (68) vanishes in the limit because of the factor z^2. In the third integral only an ε neighborhood of the point (or points) where $\phi = 0$, is critical

$$z^2 \int_{-\varepsilon}^{+\varepsilon} [\rho^{m-2} a^{m-1}(a + \cos\phi)]\frac{d\ell}{d\phi}\cos^{-1}\phi \left[\frac{z^2}{\rho^2} + \sin^2\phi\right]^{-1} d(\sin\phi)$$

The term in the first bracket is bounded. Carrying out the integration for the remaining expression one obtains as bound

$$\Big| \; z^2 \text{ const } \rho z^{-1}\text{arctg}(z^{-1}\rho\sin\phi) \Bigg|_{\phi=-\varepsilon}^{\phi=+\varepsilon}$$

The arctg function is always bounded. For $z = 0$, the values are $-\pi/2$ and $+\pi/2$. The whole expression is then $O(z)$.

For n even the contribution of the sides AB and DE of the region of integration do not cancel. Therefore, the vicinity of $\rho = 0$ must be discussed.

As before, g is defined by Eq. (56). We set again

$$g = g_1 + g_2 \tag{70}$$

but now (n even)

$$g_1(a,\phi) = \int[\cos^m\phi\sin^n\phi - a^m(1 - a^2)^{n/2}][a - \cos\phi]^{-1}d\phi$$

$$g_2(a,\phi) = a^m(1 - a^2)^{n/2}\bar{g}_2 \tag{71}$$

with

$$\bar{g}_2 = \int[a - \cos\phi]^{-1}d\phi \tag{72}$$

Incidentally, for n = 0, m = 0,

$$g = \bar{g}_2 \tag{73}$$

One verifies that

$$\bar{g}_2 = 2(a^2 - 1)^{1/2}\text{arctg}[(a + 1)^{1/2}(a - 1)^{-1/2}\text{tg}(\phi/2)]$$

Using the definition of "a" Eq. (56) one finds

$$(a + 1)^{1/2}(a - 1)^{-1/2} = (a + 1)(a^2 - 1)^{1/2} = z^{-1}(\rho + (\rho^2 + z^2)^{1/2})$$

Therefore,

$$z\bar{g}_2 = 2\rho\ \text{arctg}[z^{-1}(\rho + (\rho^2 + z^2)^{1/2})\text{tg}(\phi/2)] \tag{74}$$

$$\frac{\partial}{\partial z}(zg_2) = -\frac{-2\rho\ \text{tg}(\phi/2)[(\rho^2 + (\rho^2 + z^2)^{1/2}) - z^2(\rho^2 + z^2)^{-1/2}]}{z^2 + (\rho + (\rho^2 + z^2)^{1/2})^2\ \text{tg}^2(\phi/2)}$$

The expression Eq. (74) will be encountered only if $\rho \neq 0$. In the limit $z \to 0$ one can therefore replace $(\rho^2 + z^2)^{1/2}$ by ρ. One then obtains

$$\frac{\partial}{\partial z}(z\bar{g}_2) = \frac{-4\rho^2 \ tg\phi/2}{z^2 + 4\rho^2 \ tg^2\phi/2} \quad (z \text{ small}, \ \rho \neq 0\ ) \qquad (75)$$

Furthermore,

$$\lim_{z \to 0} \frac{\partial}{\partial z}(z\bar{g}_2) = -\cot(\phi/2)$$

Of course, the expression is meaningless for $\phi = 0$. If $\rho \neq 0$ and $\phi \neq 0$, the last equation can be obtained directly from Eq. (72) in the following manner.

$$\lim_{z \to 0}(\partial(z\bar{g}_2)/\partial z) = \lim_{z \to 0} \bar{g}_2 = \int (1 - \cos\phi)^{-1} d\phi = \int (2\sin^2(\phi/2)^{-1} d\phi$$

$$\lim_{z \to 0}(\partial(z\bar{g}_2)/\partial z) = -\cot(\phi/2)$$

For $\phi = \pi$ and $\phi = -\pi$ one has respectively $tg(\phi/2) = \infty$ and $tg(\phi/2) = -\infty$. Therefore, from Eq. (74)

$$z\bar{g}_2(\pi) = \rho\pi$$

$$z\bar{g}_2(-\pi) = -\rho\pi \qquad (76)$$

and

$$\partial(zg_2/\partial z \Big|_{\phi=\pi} = \partial(z\bar{g}_2/\partial z) \Big|_{\phi=\ -\pi} = 0$$

We begin with the contribution of g_2 to $\bar{w}_{mn}$, because there the vicinity of $\rho = 0$ requires some extra attention. To evaluate the downwash, one has to form Eq. (59). After substitution of Eq. (71) it assumes the form

$$\bar{w}_{mn,2} = \beta^2(-)^{m+n+1}\lim_{z\to 0}\frac{\partial}{\partial z}2(z\int \rho^{m+n-1}a^{m-1}(1-a^2)^{n/2}\,\bar{g}_2(a,\phi)d\rho)$$

and with the first of Eqs. (58),

$$\bar{w}_{mn,2} = \beta^2(-1)^{m+n+1+(n/2)}\lim_{z\to 0}\frac{\partial}{\partial z}(\int \rho^{m-1}a^{m-1}z^n(z\bar{g}_2(a,\phi))d\rho) \tag{77}$$

Along the portions AB and DE of the path around ABCDE, $\phi = -\pi$ and $\phi = +\pi$, respectively; $\rho(\phi)$ varies, respectively, from zero to $\rho(-\pi) = \rho(\pi)$ and from $\rho(\pi)$ to zero. Substituting Eqs. (76), one then obtains (for the two portions combined)

$$2\pi\beta^2(-)^{m+n+(n/2)}\lim_{z\to 0}[nz^{n-1}\int_0^{\rho(\pi)}\rho^m a^{m-1}d\rho + (m-1)z^{n+1}\int_0^{\rho(\pi)}\rho^{n-2}a^{m-3}d\rho]$$

The first term in the bracket vanishes for all even values of n; for $n = 0$, because of the factor $n = 0$, and otherwise because of the factor z^{n-1}.

For $m \geq 2$, the integral in the second term gives a bounded quantity. In the limit $z = 0$, the expression vanishes because of the power of z. For $m = 1$, the second term vanishes because of the factor $(m-1)$.

For $m = 0$, one goes back to the first formulation, Eq. (77). Again substituting Eq. (76), one obtains for the integral along AB

$$\pi z^n(\rho^2 + z^2)^{1/2}\Big|_0^{\rho(\pi)}$$

Forming $\lim_{z\to 0}\frac{\partial}{\partial z}$ (...) gives 0 for $n \geq 2$.

For $m = 0$, $n = 0$, one obtains

$$\frac{\pi z}{(\rho^2 + z^2)^{1/2}}\Big|_0^{\rho(\pi)} = -\pi$$

for this limit. Accordingly, the only contribution of g_2 for the portions AB and DE of the path is encountered for $m = 0$, $n = 0$, and one obtains

$$-2\pi\beta^2 \tag{78}$$

For the portion BCD (which is the map in the $\rho\phi$ plane of the contour of the $\xi\eta$-element) the expression, Eq. (77), is written in the form

$$\beta^2(-)^{m+n+1+(n/2)}\left\{\oint \rho^{m-1} z g_2 \frac{\partial}{\partial z}(z^n a^{m-1})d\rho + \oint \rho^{m-1} z^n a^{m-1} \frac{\partial}{\partial z}(\bar{z} g_2)d\rho\right\}$$

Here $\rho \neq 0$. The derivative $\partial(z^n a^{(m-1)})/\partial z$ vanishes in the lim $z \to 0$ for all m and n, ($\partial a/\partial z$ is found in Eq. (59)). In the second integral, $\lim_{z\to 0} a^{m-1} = 1$. If the integral is bounded in the limit for $n = 0$, then it will vanish for $n \geq 2$, (n even) because of the factor z^n. Already substituting Eq. (75), one obtains for $n = 0$,

$$\beta\,(-)^m \oint \rho^{m+1} \frac{4tg(\phi/2)}{z^2 + 4\rho^2 tg^2\phi/2} \tag{79}$$

For the vicinity of $\phi = 0$ the integrand behaves as ϕ^{-1}. It can be shown that one obtains the correct lim $z \to 0$ by taking in the expression

$$\beta^2(-)^m \int \rho^{m-1}\cot(\phi/2)d\rho \tag{80}$$

the principal part.

The use of the principal value in Eq. (80) is justified in the following manner. First, one excludes from the region of integration ε-neighborhoods of the points where $\phi = 0$. Outside of these neighborhoods, one can make the limiting process $z = 0$ and obtains Eq. (80). It remains to show that for every value of z, the contributions of these neighborhoods tend to zero as $\varepsilon \to 0$. For this purpose we set the residue is a_{-1}. Furthermore,

$$\log(\phi_2) = \log|\phi_2|$$

$$\log(\phi_1) = \log|\phi_1| + i\pi$$

Denoting by P the principal value one finds

$$P\int_{\phi_1}^{\phi_2}(a_{-1}\phi^{-1} + a_0 + a_1\phi)d\phi = a_{-1}[\log|\phi_2| - \log|\phi_1| + a_0(\phi_2 - \phi_1)$$

$$+ a_1(\phi_2^2 - \phi_1^2)]/2$$

In every situation of this kind the integral will contain a logarithm. The contribution of the residue is automatically taken into account by taking the absolute values of logarithms.

This makes an additional step possible. In the integral in the complex plane

$$w_{m,0,2}\Big|_{\phi_1}^{\phi_2} = \beta^2(-)^m \int_{\phi_1}^{\phi_2} \rho^{m-1} \cot(\phi/2)(d\rho/d\phi)d\phi$$

an integration by parts is carried out. Then

$$w_{m,0,2}\Big|_{\phi_1}^{\phi_2} = \beta^2(-)^m\Bigg[m^{-1}\Bigg[\rho^m \cot(\phi/2)\Big|_{\phi_1}^{\phi_2}$$

$$+ \int_{\phi_1}^{\phi_2} \frac{\rho^m}{1 - \cos\phi}\Bigg]d\phi - i\pi \text{ residue}\Bigg]$$

Taking the above example and carrying an integration by parts one obtains

$$\int_{\phi_1}^{\phi_2} (a_{-1}\phi^{-1} + a_0 + a_1\phi)d\phi$$

$$= \phi(a_{-1}\phi^{-1} + a_0 + a_1\phi)\Big|_{\phi_1}^{\phi_2} - \int_{\phi_1}^{\phi_2} (a_1^{-1}\phi^{-1} + a_1\phi)d\phi$$

The integrand after the integration by parts has the same residue as the original integral. Again, one takes in the analytic expression for the resulting integral the absolute value of the logarithm.

If one carries out the integration only along a portion of the contour, then one must of course include the expression outside of the integral. If one makes this integration by parts along all sections of the contour, then the terms outside of the integral cancel each other.

The same integration by parts can, of course, be made for a contour of the $\xi\eta$-element for which $\phi \neq 0$ everywhere. Thus one can always write

$$\bar{w}_{m,0,2} = \beta^2(-)^m \oint \rho^m(1 - \cos\phi)^{-1}d\phi \quad (n = 0),\ m \geq 1 \tag{81}$$

if one uses for the individual sections of the contour, analytic expressions for the integrals, substitutes the limits, and takes the absolute values of the logarithms which may occur.

For $m = 0$, $n = 0$, the evaluation of the expression Eq. (80), now given by

$$\beta^2 \oint \rho^{-1} \cot(\phi/2)d\rho$$

is carried out as follows. One has

$$\cot(\phi/2) = (1 + \cos\phi)/\sin\phi = -(1 - \cos\alpha)/\sin\alpha$$

We defined $\xi - x = \rho \cos\alpha$, $\eta - y = \rho \sin\alpha$. Accordingly,

$$\iint \rho^{-1} \cot(\phi/2) d\rho = - \oint \frac{d\rho}{\eta - y} + \oint \frac{(\xi - x)\rho d\rho}{\rho^2(\eta - y)}$$

The second integrand is transformed into

$$(\xi - x)[(\xi - x)d\xi + (\eta - y)d\eta]/(\rho^2(\eta - y))$$

$$= [(\xi - x)^2 d\xi + (\xi - x)(\eta - y)d\eta]/(\rho^2(\eta - y))$$

Here $(\xi - x)^2$ is replaced by $\rho^2 - (\eta - y)^2$.

One then obtains

$$(d\xi/(\eta - y)) + ([(\xi - x)d\eta - (\eta - y)d\xi]/\rho^2) = (d\xi/\eta - y)) + d\phi$$

For points xy outside or inside the $\xi\eta$-element, one has, respectively, $\oint d\phi = 0$ and $\oint d\phi = 2\pi$. The second expression cancels the contribution to $\bar{w}_{0,2}$ *shown in* Eq. (78). Moreover $g_1 = 0$ for $m = 0$, $n = 0$. Thus one obtains, generally,

$$\bar{w}_{00} = \beta^2 \oint \frac{(d\xi/d\ell) - (d\rho/d\ell)}{\eta - y} d\ell \qquad (82)$$

For $(\eta - y) = 0$,

$$d\rho = d\xi \operatorname{sign}(\xi - x)$$

If $\eta - y = 0$ and $(\xi - x) < 0$ (x in the wake of ξ), a singularity will arise and the expression must be interpreted (according to the above discussion) as principal part. In the discussion following Eq. (78) we found that

$$\bar{w}_{m,n,2} = 0 \qquad \text{for even } n \geq 2 \qquad (83)$$

The portions AB and DE of the contour in the ρ,ϕ plane start at $\rho = 0$. There, $\phi = -\pi$ and $+\pi$, respectively, and one has to form according to Eq. (59)

$$\lim_{z\to 0} \frac{\partial}{\partial z}\left\{z(-)^{m+n+1}\beta^2 \int_0^{\rho(-\pi)} \rho^{m+n-1} a^{-1}[g_1(a,-\pi) - g_1(a,\pi)]d\rho\right\}$$

$$= \beta^2(-)^{m+n-1} \lim_{z\to 0} \left\{\int_0^{\rho(-\pi)} \rho^{m+n-1} a^{-1}[g_1(a,-\pi) - g_1(a,\pi)]d\rho\right.$$

$$\left. + z^2 \int_0^{\rho(-\pi)} \rho^{m+n-3} a^{-1} \frac{\partial}{\partial a}(a^{-1}[g_1(a,-\pi) - g_1(a,\pi)])d\rho\right\}$$

Forming the limit $z\to 0$ one replaces in the first integral, a by 1. The second integral vanishes because of the factor z^2 if $m+n \geq 3$. We are discussing cases for $m+n \geq 1$ and n even. Since n is even, $m+n=1$ implies $m = 1$. Then by its definition, Eq. (71)

$$g_1 = \int (\cos\phi - a)(a - \cos\phi)^{-1} d\phi = -\phi$$

and one has to form

$$\lim_{z\to 0} \frac{\partial}{\partial z}\left(z \int_0^{\rho(-\pi)} a^{-1} d\rho\right) = \lim_{z\to 0} \frac{\partial}{\partial z}(\rho^2 + z^2)^{1/2} \Big|_0^{\rho(-\pi)}$$

$$= \lim_{z\to 0} \left[(\rho^2 + z^2)^{1/2} + \frac{z^2}{(\rho^2 + z^2)^{1/2}}\right]\Bigg|_0^{\rho(-\pi)} = \rho(-\pi)$$

This is the same as if one had set $a = 1$ in the $\int a^{-1} d\rho$. For $m+n = 2$, one has either $n = 2$, $m = 0$, or $m = 0$, $n = 2$. In each case one deals (except for the sign) with an integral

$$g_1(a,\phi) = \int \frac{\cos^2\phi - a^2}{\cos\phi - a} d\phi = \int (\cos\phi + a)d\phi = \sin\phi + a\phi$$

Then $g_1(a,\pi) - g_1(a,-\pi) = 2\pi a$ and one has to form

$$\lim_{z\to 0} \frac{\partial}{\partial z} (z \int \rho 2\pi d\rho) = \pi\rho^2(\pi)$$

One thus obtains in all cases as combined contribution of the portions AB and DE due to g_1

$$\beta^2(-1)^{m+n+1}(g_1(1,-\pi) - g_1(1,+\pi)(m+n)^{-1}\rho^{m+n}(\pi)$$

This expression cancels the term outside the integral in Eq. (85). One thus obtains (after returning to α)

$$\bar{w}_{m,n} = \beta^2(m+n)^{-1} \int \rho^{m+n}\cos^m\alpha \sin^n(\alpha)[1 + \cos\alpha]^{-1}d\alpha, \qquad (86)$$

$$\text{even } n \geq 2$$

Since, according to Eq. (83) $\bar{w}_{mn,2} = 0$ for $n \geq 2$, this is $\bar{w}_{mn}$ (rather than $\bar{w}_{mn,1}$). Moreover

$$\bar{w}_{m,0,1} = \beta^2 m^{-1} \int \rho^m(\cos^m\alpha - (-1)^m)[1 + \cos\alpha]^{-1}d\alpha \quad , \begin{matrix} n = 0 \\ m \geq 1 \end{matrix}$$

This is combined with the expression $\bar{w}_{m,0,2}$, Eq. (81). One then obtains

$$\bar{w}_{m,0} = \bar{w}_{m,0,1} + \bar{w}_{m,0,2} = \beta^2 m^{-1} \oint \rho^m\cos^m\alpha[1 + \cos\alpha]^{-1}d\alpha, \qquad (86a)$$

if, as was stated above, one uses for the individual sections of the contour analytic expressions for the integral, substitutes the limits and takes the absolute value of the logorithms, if they should occur. Actually, with this interpretation Eq. (86) is generally valid.

These results give the contribution to K obtained for the power zero of the frequency k, which will be indicated by a superscript. In the original coordinates one obtains

$$\bar{w}^0_{mn}(xy) = \beta^2(m+n)^{-1} \oint (\xi-x)^m(\eta-y)^n[\rho(\rho+(\xi-x))]^{-1}[(\xi-x)d\eta-(\eta-y)d\xi] \tag{87}$$

except for $m = 0$, $n = 0$

For n odd, this is a repetition of Eq. (69). For $n = 0$, one must follow the procedure described after Eq. (86a).

Furthermore,

$$\bar{w}^0_{00}(x,y) = \beta^2 \oint (d\xi - d\rho)(\eta-y) \tag{88}$$

Eq. (51) gives the result for the first power of the reduced frequency. Introducing again the form Eq. (52) for $\Delta\bar{p}$ considering a single term and carrying out the integration with respect to ρ, one obtains

$$w^1_{mn} = (m+n+1)^{-1} \int(\xi-x)^m(\eta-y)^{m-1}[(\xi-x)d\eta-(\eta-y)d\xi] \tag{88a}$$

The results of these discussions can be summarized in a simple manner. We assume that for x,y fixed, $\Delta\bar{p}$ has the form of Eq. (51), i.e., a development in powers of $(\xi-x)$ and $(\eta-y)$. Except for $m = 0$, $n = 0$, the correct expression for the upwash at $z = 0$ is obtained by setting $z = 0$ in the expression for K. One then has to evaluate integrals

$$I = \iint F(\xi-x,\eta-y)d\xi d\eta \tag{89}$$

For $z = 0$, one has $R = \rho$. Let

$$\xi-x = R\cos\alpha$$
$$\eta-y = R\sin\alpha$$

For one of the summands in $\Delta\bar{p}$ and one power of the reduced frequency k, the function F has the form

$$F = R^q f(\alpha) \tag{90}$$

Then

$$I = \iint R^q f(\alpha) R dR d\alpha$$

$$= (q+2)^{-1} \oint R(\alpha)^{q+2} f(\alpha) d\alpha$$

or in terms of the original coordinates

$$I = (q+2)^{-1} \oint F(\xi,\eta)[(\xi-x)d\eta-(\eta-y)d\xi] \tag{91}$$

If $q = -2$ (i.e, for $m = 0$, $n = 0$ and for the power zero of the reduced frequency), the procedure is not feasible. The integration with respect to R gives log R. The limits for R are 0 and $R(\alpha)$. The lower limit gives infinity, which is an indication that this simple minded procedure fails. Here it is necessary to proceed in the manner described above; one carries out limiting process $z \to 0$ only after the integrations with respect to ξ and η (or ρ and α) have been carried out. This gives Eq. (88). Another limitation of this rule arises if $n = 0$, $m \geq 1$, the power of the reduced frequency is zero and if along the contour $\eta-y$ becomes zero. Then the integrand will contain a denominator $(\eta-y)^2$, and the integral becomes nonsensical. One then has to apply the procedure described after Eq. (86a).

In K_1, one finds terms which, after one sets $z = 0$, gives rise to integrals

$$I = \iint \log(\eta-y) F(\xi-,\eta-y) d\xi d\eta \tag{92}$$

where F has the form assumed above. Then

$$I = \iint \log(R\sin\alpha)R^{q+1}f(\alpha)dRd\alpha$$

$$= \iint (\log R + \log \sin\alpha)R^{q+1}f(\alpha)dRd\alpha$$

$$= (q+2)^{-1} \int f(\alpha)R^{q+2}[\log R(\alpha) - (q+2)^{-1} + [\log \sin\alpha)]d\alpha$$

This is written in terms of the original coordinates

$$I = (q+2)^{-1} \oint F(\xi,\eta)[\log(\eta-y) - (q+2)^{-1}][(\xi-x)d\eta - (\eta-y)d\xi] \quad (93)$$

SECTION V
THE UPWASH AT A GIVEN POINT (x,y)

We evaluate in this section the upwash at a fixed point x,y due to a pressure distribution in a given $\xi\eta$-element. To each triangular or trapezoidal elements belong, respectively, 3 or 4 elemental pressure distributions. The results can be expressed in terms of elementary transcendental functions, provided that the elemental pressure distributions have the form of polynomials in ξ and η. These formulae will be needed if (x,y) lies within or in the vicinity of a $\xi\eta$-element. For triangular element and $\Delta\bar{p}$ linear in ξ and η, one obtains pressure distributions which are continuous at the common boundary of neighboring elements. For trapezoidal elements, one can achieve only imperfect matching even if one uses terms of higher order in ξ and η. Except for this flaw in the basic data, the analysis is independent of the shape of the elements.

For triangular elements the corner opposite the side that is parallel to the ξ axis has the number 1, from thereon one proceeds in the counterclockwise direction. The elemental pressure distributions are then given by

$$\Delta\bar{p}^{(i)}(\xi,\eta) = c_0^{(i)} + c_1^{(i)}(\xi-\xi_1) + c_2^{(i)}(\eta-\eta_1) \qquad i = 1,2,3$$

where

$$\begin{aligned}
&c_0^{(1)} = 1,\; c_1^{(1)} = 0,\; c_1^{(2)} = -(\eta_2-\eta_1)^{-1} \\
&c_0^{(2)} = 0,\; c_1^{(2)} = D^{-1}(\eta_3-\eta_1),\; c_2^{(2)} = -D^{-1}(\xi_3-\xi_1) \\
&c_0^{(3)} = 0,\; c_1^{(3)} = -D^{-1}(\eta_2-\eta_1),\; c_2^{(3)} = D^{-1}(\xi_2-\xi_1) \\
&D = [(\xi_2-\xi_1)(\eta_3-\eta_1) - (\xi_3-\xi_1)(\eta_2-\eta_1)]
\end{aligned} \qquad (94)$$

The upwash is evaluated separately for these three expressions $\Delta\bar{p}^{(i)}$.

We shall write during the derivations of the formulae

$$\Delta\bar{p}(\xi,\eta) = \Sigma c_{mn}^{(i)}(x,y)(\xi-x)^m(\eta-y)^n$$

For triangular elements one has in particular

$$\Delta\bar{p}(\xi,\eta) = c_{oo}^{(i)}(x,y) + c_{10}^{(i)}(\xi-x) + c_{01}^{(i)}(\eta-y) \qquad (95)$$

where

$$c_{00}^{(i)}(x,y) = c_0^{(i)} + c_1^{(i)}(x-\xi_1) + c_2^{(i)}(\eta-y_1)$$

$$c_{10}^{(i)} = c_1^{(i)} \; ; \; c_{01}^{(i)} = c_2^{(i)}$$

Later, in integrations over x and y, each of the expressions

$$\Delta\bar{p}^{(i)}(x,y) = c_{00}^{(i)}(x,y) + c_{10}^{(i)}(\xi-x) + c_{01}^{(i)}(\eta-y)$$

will be used in three different forms.

$$\Delta\bar{p}^{(i)}(x,y) = \delta_{ij} + c_{10}^{(i)}(x-\xi_j) + c_{01}^{(i)}(y-\eta_j) \qquad (96)$$

Here the subscript i refers to those corners of the $\xi\eta$-element for which the elemental pressure is 1. (For the other corners the elemental pressure is zero); j is the subscript of any of the three corners when it is used as origin of a x,y-system.

The discussion of the limiting process $z \to 0$ in Section V has led to integrals around the contour of the $\xi\eta$-element which give the upwash at a fixed point x,y due to a pressure distribution $\Delta\bar{p}^{(i)}(\xi,\eta)$. If the point xy lies in the interior of the $\xi\eta$-element or of its neighbors then the integrands of these expressions are analytic functions except for well defined singularities. The author hoped originally that the smooth part of these integrals could be evaluated numerically. But upon closer examination, he realized that for a point (xy) close to the boundary of the $\xi\eta$-element the integration over ξ and η will encounter difficulties in spite of the analyticity of the integrands. Fortunately, the integrals can be evaluated in terms of elementary transcendental functions, although one obtains fairly lengthy expressions.

The difficulties arise in the following manner. The integrands are analytic for real values of the variable of integration, but in some vicinity of the origin they have singularities (poles and branch points) at complex values. As the point (xy) moves closer to a boundary of the $\xi\eta$ element the interval of integration extends farther and farther along the real axis (if one keeps the singularities in fixed positions). One might then divide the region of integration into a section close to the singularities and a remaining part. But even the second section is not well suited for numerical integrations. Let z be the variable of integration and f(z) the integrand. A suitable variable of integration in the outer region would then be 1/z and one must form

$$- \int z^2 f(z) d(1/z)$$

where (1/z) is the variable of integration. This works well only if $z^2 f(z)$ is a regular function of (1/z) for $1/z = 0$. The terms which violate this requirement are fairly easily identified, so that one could treat them separately. But since an analytical integration over the whole range is possible it appears more practical to follow this course. Naturally, once programmed one will use these formulae also if the point (x,y) is not close to the element boundaries. Later this procedure will make it possible to carry out also the integration over x and y analytically. The integrals to be evaluated are given by

$$\iint \Delta p(\xi-\eta) K(\xi-x,\eta-y,k) d\xi d\eta$$

The function K is found in Eq. (B.63)

$$K = \beta^{-1} \exp(ik(\xi-x)[K_1(\xi-x,r,k) + K_2(\xi-x,r,k)]$$

where, according to Section IV, r can be replaced in most cases by $|\eta-y|$.

We have introduced

$$\Delta\bar{p}(\xi,\eta) = \exp(ik\xi)\Delta p(\xi,\eta) \qquad (97)$$

Within the elements $\Delta\bar{p}(\xi,\eta)$ is represented by shape functions which are linear in ξ and η. The infinite sums in K_2 can be truncated after only very few terms, because k is usually small and because the present discussion refers to points (x,y) within or close to the $\xi\eta$-element. In the following term of K_2 the exponential function is expressed as a power series.

$$(RV)^{-1}(\exp(ikV) - 1) = ikR^{-1} \sum_{1}^{\infty} (ikV)^{\ell-1}/\ell!$$

In Ueda's formulation one finds a similar term namely $R(V^2 + r^2)^{-1/2}$ $\exp(ikV)$. The present simpler form arises from a combination of this term with other expressions of K, (see the discussion following Eq. (B.59). In the present context this brings about a considerable simplification in the integrations. In the procedure of Ueda and Dowell the simplification is only minor. All terms of K_2 can now be brought into a form

$$P(\xi-x,\eta-y) + R^{-1} P(\xi-x,\eta-y)$$

where $P(\xi-x,\eta-y)$ denotes a generic expression for a power series in their arguments. In practice one deals with polynomials.

We shall derive integration formulae for the lowest powers of the reduced frequency k of the development of K, namely k^o, k, $k^2\log k$ and k^2. For a portion of $\Delta\bar{p}$ given by $(\xi-x)^m(\eta-y)^n$, we shall denote by $\bar{I}^{\ell}{}_{mn}$ the integral over the $\xi\eta$-element. The superscript ℓ refers to the power of the reduced frequency k. $\bar{I}^{\ell}_{mn}$ does not contain the factor $(8\pi)^{-1}\beta^{-1}\exp(-ikx)$. After substituting Eqs. (B.66) and (B.67) one obtains the following expression

$$\bar{I}^0_{mn} = \beta^2 \iint (\xi-x)^m(\eta-y^n)r^{-2}R^{-1}(R - (\xi-x))d\xi d\eta; \quad m+n \geq 1 \tag{98}$$

(The case m = 0, n = 0 has been treated separately.)

$$\bar{I}^1_{mn} = -ik \iint (\xi-x)^m(\eta-y)^n R^{-1} d\xi d\eta; \quad m+n > 0 \tag{99}$$

$$\bar{I}^2_{mn} = \frac{k^2}{2} \iint (\xi-x)^m(\eta-y)^n[C - (\beta^{-2}(\xi-x)R^{-1}) + (\log(k(R + (\xi-x)))]d\xi d\eta \tag{100}$$

Here,

$$r^2 = (\eta-y)^2, \; R = [(\xi-x)^2 + (\eta-y)^2]^{1/2} \tag{101}$$

The function V which occurs in Eq. (B.67) is found in one of the Eqs. (B.61).

$$C = -(1/2)[\psi(1) + \psi(2) + \log(1-M) + 2\log 2 - i\pi + \beta^{-2}M] \tag{102}$$

Then, according to Eqs. (89), (91), (92), and (33)

$$\bar{I}^0_{mn} = \beta^2(m+n)^{-1}\oint(\xi-x)^m(\eta-y)^{n-2}R^{-1}(R-(\xi-x))[(\xi-x)d\eta-(\eta-y)d\xi] \quad (103)$$

$$\bar{I}^1_{mn} = -ik(m+n+1)^{-1}\oint(\xi-x)^m(\eta-y)^nR^{-1}[(\xi-x)d\eta-(\eta-y)d\xi] \quad (104)$$

$$\bar{I}^2_{mn} = (k^2/2)(m+n+2)^{-1}\oint(\xi-x)^m(\eta-y)^n\{C-(\beta^{-2}(\xi-x)R^{-1}) + (\log(k(R+(\xi-x)) - (m+n+2)^{-1})[(\xi-x)d\eta - (\eta-y)d\xi] \quad (105)$$

The path of integration is the contour of the element. Practically this means that we have separate expressions for the sides of the triangular or trapezoidal element. Accordingly we write

$$\bar{I}^\ell_{mn} = {}^{12}\bar{I}^\ell_{mn} + {}^{23}\bar{I}^\ell_{mn} + {}^{31}\bar{I}^\ell_{mn} \quad (106)$$

where the left superscripts refer to the end points of the sides (as one travels around the $\xi\eta$-element in the counterclock-wise direction). We shall derive separate formulae for the sides parallel to the ξ axis and sides inclined to the ξ axis. For a side of the $\xi\eta$-element between the point i and i+1, not parallel to the ξ-axis, (where for triangles i+1 = 4 is understood to refer to point 1), we define

$$tg\alpha_{i,i+1} = (\xi_{i+1} - \xi_i)/(\eta_{i+1} - \eta_i) \quad (107)$$

The subscripts of α will frequently be omitted. The expressions $\bar{I}$ appear first as indefinite integrals. After substitution of the limits of integration (expressed by ξ_i, η_i, and ξ_{i+1}, η_{i+1}), one obtains two expressions $(x-\xi_i,\ y-y_i)$ and $(x-\xi_{i+1}, y-\eta_{i+1})$. Those expressions will be denoted by

$$I^\ell_{mn}(x-\xi_i,\ y-\eta_i,\ \alpha_{i,i+1}) \text{ and } I^\ell_{mn}(x-\xi_{i+1},\ y-\eta_{i+1},\ \alpha_{i,i+1})$$

so that

$$ {}^{i,i+1}\bar{I}^{\ell}_{mn} = I^{\ell}_{mn}\,(x-\xi_{i+1}, y-\eta_{i+1}, \alpha_{i,i+1}) - I^{\ell}_{mn}\,(x-\xi_i, y-\eta_i, \alpha_{i,i+1}) \quad (108) $$

Information about the side for which the expression is being evaluated enters through the argument $\alpha_{i,i+1}$.

For the side parallel to the ξ axis, we insert $\pi/2$ as argument of α. The expressions $I(..\alpha)$(to be found later) tend to infinity as $\alpha \to \pi/2$, but $I(..\pi/2)$ remains finite. This happens because of the occurrence of α-dependent integration constant which tends to infinity as $\alpha \to \pi/2$. They cancel as one forms the integral between points i and i+1. The expressions $R = ((\xi-x)^2 + (\eta-y)^2)^{1/2}$ will introduce branch points of the second order in the integrands. We write

$$ I^{\ell}_{m,n} = I^{\ell,1}_{mn} + I^{\ell,2}_{m,n} \quad (109) $$

The additional superscripts 1 and 2 refer respectively to the absence and presence of such branch points. For $\alpha = \pi/2$, one has $\eta = \eta_i = \eta_{i+1} = \text{const}$, $d\eta = 0$. Then we shall introduce,

$$ \eta_i - y = v \quad (110) $$

$$ (\xi - x) = w.v \quad (111) $$

and consider w as variable of integration. For the power zero of the reduced frequency, one has according to Eq. (90)

$$ I^{0}_{00} = +\beta^2 \left[\iint \frac{d\xi}{\eta-y} - \int \frac{dR}{\eta-y}\right] \quad (112) $$

R is defined in Eq. (101). For $\alpha_{ij} = \pi/2$ one has to evaluate (because $d\eta = 0$).

$$ {}^{i,i+1}I^{0}_{00} = {}^{i,i+1}I^{01}_{00} + {}^{0,i+1}I^{02}_{00} $$

with

$$ {}^{i,i+1}I_{00}^{01} = + \beta^2 \int_{\xi_i}^{\xi_{i+1}} \frac{d\xi}{(\eta_i - y)} $$

$$ {}^{i,i+1}I_{00}^{02} = -\beta^2 \int_{\xi_i}^{\xi_{i+1}} \frac{(\xi - x)d\xi}{R(\eta_i - y)} $$

Hence,

$$ {}^{i,i+2}I_{00}^{02} = \left(\beta^2 \frac{\xi}{\eta_i - y}\right) $$

and with Eq. (111),

$$ {}^{i,i+1}I_{00}^{02} = -\beta^2 \, \mathrm{sign}(\eta_i - y) \int_{w_i}^{w_{i+1}} w(w^2 + 1)^{-1/2} dw $$

$$ = -\beta^2 \mathrm{sign}(\eta_i - y)(w^2 + 1)^{1/2} \Big|_{w_i}^{w_{+1}} $$

If one goes back to the original variables, then one obtains

$$ (w^2 + 1)^{1/2} = \mathrm{sign}(\eta_i - y)R/(\eta_i - y) $$

because R is always positive. The sign functions cancel in the original coordinates. This happens in all formulae. For simplicity we shall omit in future computations such intermediate sign expressions. For future work it is convenient to introduce

$$X = x - \xi_i, \; Y = y - \eta_i \tag{113}$$

and

$$U = -u, \; V = -V$$

Then,

$$I_{00}^{01}(X,Y,\pi/2) = \beta^2 X/Y \tag{114}$$

$$I_{00}^{02}(X,Y,\pi/2) = \beta^2 R/Y \tag{115}$$

According to Eq. (103) one has to evaluate for $\alpha = \pi/2$, the power zero of the reduced frequency k, and $m+n \geq 1$

$$^{i,i+1}\bar{I}_{mn}^{0} = {}^{i,i+1}\bar{I}_{mn}^{01} + {}^{i,i+1}\bar{I}_{mn}^{02}$$

with

$$^{i,i+1}\bar{I}_{mn}^{01} = -\beta^2(m+n)^{-1}(\eta_i - y)^{n-1} \int_{\xi_i}^{\xi_{i+1}} (\xi-x)^m d\xi$$

$$= -\beta^2(m+n)^{-1}(m+1)^{-1}(\eta_i - y)^{n-1}(\xi-x)^{m+1} \Big|_{\xi_0}^{\xi_{i+1}}$$

and

$$^{i,i+1}\bar{I}_{mn}^{0,2} = \beta^2(m+n)^{-1}(\eta_i - y)^{n-1} \int_{\xi_i}^{\xi_{i+1}} (\xi-x)^{m+1} R^{-1} d\xi$$

With

$$w = (\xi - x)/(\eta_i - y)$$

one obtains

$$^{i,i+1}\bar{I}^{0,2}_{mn} = \beta^2(m+n)^{-1}(\eta_i - y)^{m+n} \int_{w_i}^{w_{i+1}} (w^2 + 1)^{-1/2} w^{m+1} dw$$

Formulae for the integrals occurring here are found in Appendix C. Specializing to [m,n] = [1,0] and [0,1] one obtains

$$I^{01}_{10}\ (X,Y,\pi/2) = (\beta^2/2)X^2/Y$$

$$I^{02}_{10}\ (X,Y,\pi/2) = (\beta^2/2)[(RX/Y) + Y\log(k(R - X))] \qquad (116)$$

$$I^{01}_{01}\ (X,Y,\pi/2) = \beta^2 X$$

$$I^{02}_{01}\ (X,Y,\pi/2) = \beta^2 R$$

The factor k in the logarithm amounts to a change of the constant of integration. If the expression is written in this form, it will not change if one changes the reference length L.

For $\alpha = \pi/2$ and the power 1 of the reduced frequency k one obtains from Eq. (104)

$$^{i,i+1}\bar{I}^{1}_{mn} = + ik(m+n+1)^{-1} \int_{\xi_i}^{\xi_{i+1}} R^{-1}(\xi - x)^m(\eta_i - y)^{n+1} d\xi$$

Again, $w = (\xi - x)/\eta_i - y)$ is introduced and one obtains

$$^{i,i+1}\bar{I}^{1}_{mn} = ik(m+n+1)^{-1}(\eta_i - y)^{m+n+1} \int_{w_i}^{w_{i+1}} (w^2 + 1)^{-1/2} w^m dw$$

These integrals are found in Appendix C.

$$\int (w^2 + 1)^{-1/2} dw = \log ((w^2 + 1)^{1/2} + w)$$

$$\int w(w^2 + 1)^{-1/2} dw = (w^2 + 1)^{1/2}$$

One obtains

$$I^{1}_{00}(X,Y,\pi/2) = -ikY\log(k(R - X))$$

$$I^{1}_{10}(X,Y,\pi/2) = -i(k/2)YR \tag{117}$$

$$I^{1}_{01}(X,Y,\pi/2) = i(k/2)Y^2\log(k(R - X)$$

For the powers k^2 and $k^2 \log k$ one evaluates according to Eq. (86).

$$\bar{I}^{2}_{m,n} = \bar{I}^{2,1}_{m,n} + \bar{I}^{2,2}_{m,n} + \bar{I}^{2,3}_{m,n}$$

with

$$^{i,i+1}\bar{I}^{21}_{mn} = -(k^2/2)(m+n+2)^{-1}C \int_{\xi_i}^{\xi_{i+1}} (\xi - x)^m (\eta_i - y)^{n+1} d\xi$$

$$^{i,i+1}\bar{I}^{2,2}_{mn} = (k^2/2)(m+n+2)^{-1} \int_{\xi_i}^{\xi_{i+1}} (\xi - x)^{m+2} (\eta_i - y)^{n+1} R^{-1} d\xi$$

$$ {}^{i,i+1}\bar{I}^{23}_{mn} = $$

$$ -(k^2/2)(m+n+2)^{-1} \int_{\xi_i}^{\xi_{i+}} (\xi-x)^m(\eta_i - y)^{n+1}[\log(k(R + \xi-x))-(m+n+2)^{-1}]d\xi $$

Hence

$$ I^{21}_{00} (X,Y,\pi/2) = -(k^2/4)CXY $$

$$ I^{21}_{10} (X,Y,\pi/2) = (k^2/12)CX^2Y \qquad (118) $$

$$ I^{21}_{01} (X,Y,\pi/2) = + (k^2/6)CXY^2 $$

In I^{22}_{mn} one introduces $w = (\xi-x)/(\eta_i - y)$ in an intermediate step and thus obtains integrals listed in Appendix C. Hence

$$ I^{22}_{00} (X,Y,\pi/2) = -(k^2/4)\beta^{-2}YR $$

$$ I^{22}_{10} (X,Y,\pi/2) = (k^2/12)\beta^{-2}[YXR + Y^3\log(k(R - X)] \qquad (119) $$

$$ I^{22}_{01} (X,Y,\pi/2) = (k^2/6)\beta^{-2}Y^2R $$

Furthermore,

$$ {}^{i,i+1}\bar{I}^{2,3}_{mn} = -2(k/2)^2(m+n+2)^{-1}(\eta_i - y)^{n+1}I $$

with

$$ I = \int_{\xi_i}^{\xi_{i+1}} (\xi-x)^m[\log(k(R + \xi-x)) - (m+n+2)^{-1}]d\xi $$

Then

$$I = (m+1)^{-1}\{(\xi-x)^{m+1}[\log(k(R + \xi-x)) - (m+n+2)^{-1}]$$

$$- \int_{\xi_i}^{\xi_{i+1}} (\xi-x)^{m+1}R^{-1}d\xi\}$$

With $w = (\xi-x)/(\eta_i - y)$ one obtains for the last integral expressions listed in Appendix C; in particular

$$\int (\xi-x)R^{-1}d\xi = R$$

$$\int (\xi-x)^2R^{-1}d\xi = (1/2)\{(\xi-x)R-(\eta_i - y)^2\log(k(R + \xi-x))\}$$

Therefore

$$I_{00}^{23} = -(k/2)^2(\eta_i - y)\{(\xi-x)[\log(k(R + \xi-x))-1/2] - R\}$$

or

$$I_{00}^{23} (X,Y,\pi/2) = -(k/2)^2\{YX[\log(k(R - X)) - (1/2)] + YR\} \tag{120}$$

$$I_{10}^{23} (X,Y,\pi/2) = (1/3)(k/2)^2\{YX^2[\log(k(R - X))-1/3]$$

$$+ (1/2) XYR + (1/2)Y^3\log(k(R - X))\}$$

$$I_{01}^{23} (X,Y,\pi/2) = (2/3)(k/2)^2\{Y^2X[\log(k(R - X))-1/3] + Y^2R\}$$

This terminates the case $\alpha = \pi/2$.

For $\alpha \neq \pi/2$ auxiliary systems of coordinates are used. The "sheared" coordinate system is defined by

$$u_s = (\xi - x) - (\eta - y)\text{tg}\alpha, \quad \xi - x = u_s + v_s \text{ tg}\alpha \qquad (121)$$

$$v_s = \eta - y, \qquad \eta - y = v_s$$

Along the side i,i+1 of an element one has

$$u_s = u_{si} = u_{si+1} = \text{const.} = (\xi_i - x) - (\eta_i - y)\text{tg}\alpha \qquad (122)$$

Note that

$$(\xi - x)d\eta - (\eta - y)d\xi = u_s dv_s - v_s dus = u_s dv_s \qquad (123)$$

(The last simplification occurs because u_s= const because along the side i,i+1, and therefore $du_s = 0$.)

The "rotated" system of coordinates is defined by

$$\begin{aligned} u &= (\xi - x)\cos\alpha - (\eta - y)\sin\alpha, & \xi - x &= u \cos\alpha + v \sin\alpha \\ v &= (\xi - x)\sin\alpha + (\eta - y)\cos\alpha, & \eta - y &= -u \sin\alpha + v \cos\alpha \end{aligned} \qquad (124)$$

Along the side i,i+1 of an element one has

$$u = u_i = u_{i+1} = \text{const} \qquad (125)$$

$$u_{si} = \cos^{-1}\alpha \; u_i \qquad (126)$$

Note that

$$R^2 = (\xi - x)^2 + (\eta - y)^2 = u^2 + v^2 \qquad (127)$$

Therefore,

$$(\xi-x)d\xi + (\eta-y)d\eta = u\,du + v\,dv = v\,dv \tag{128}$$

The last simplification occurs because u = const along a side $i,i+1$. Moreover

$$(\xi-x)d\eta - (\eta-y)d\xi = u\,dv - v\,du = u\,dv \tag{129}$$

We shall furthermore introduce

$$w = v/u_i \tag{130}$$

and

$$\theta = \mathrm{arctg}\,w = \mathrm{arctg}(v/u_i) \tag{131}$$

Then

$$\begin{aligned} \sin\theta &= v/R \\ \cos\theta &= u_i/R \end{aligned} \tag{132}$$

Hence,

$$\mathrm{sign}(\cos\theta) = \mathrm{sign}\ u_i \tag{133}$$

Moreover,

$$\begin{aligned} \sin(\theta-\alpha) &= (v\cos\alpha - u_i\sin\alpha)/R = (\eta-y)/R \\ \cos(\theta-\alpha) &= (u_i\cos\alpha + v\sin\alpha)/R = (\xi-x)/R \end{aligned} \tag{134}$$

The basic formula for $\alpha \neq \pi/2$ and the power zero of the reduced frequency is quoted in Eq. (112). We introduce the rotated system of coordinates Eqs. (124) through (129) and Eq. (130). Writing

$$^{i,i+1}\bar{I}^{0}_{00} = {}^{i,i+1}\bar{I}^{01}_{00} + {}^{i,i+1}\bar{I}^{02}_{00}$$

one obtains

$$^{i,i+1}\bar{I}^{01}_{00} = \beta^2 \int_{w_i}^{w_{i+1}} \sin\alpha(w \cos\alpha - \sin\alpha)^{-1} dw \tag{135}$$

$$^{i,i+1}\bar{I}^{02}_{00} = - \beta^2 \operatorname{sign} u_i \int_{w_i}^{w_{i+1}} (w\cos\alpha - \sin\alpha)^{-1}(w^2 + 1)^{-1/2} w dw \tag{136}$$

Eq. (135) gives

$$^{i,i+1}\bar{I}^{01}_{00} = \beta^2 \operatorname{tg}\alpha \log(w-\operatorname{tg}\alpha) \Big|_{w_i}^{w_{i+1}}$$

Introducing θ, Eq. (131), one obtains

$$^{i,i+1}\bar{I}^{01}_{00} = \beta^2 \operatorname{tg}\alpha \log (\sin(\theta-\alpha)/\cos\theta \Big|_{\theta_i}^{\theta_{i+1}} \tag{137}$$

In Eq. (136), θ is introduced immediately. One observes that

$$(w^2 + 1)^{1/2} = \operatorname{sign}(\cos\theta)\cos^{-1}\theta$$

Then with Eq. (133)

$$^{i,i+1}\bar{I}_{00}^{02} =$$

$$-\beta^2 \int_{\theta_i}^{\theta_{i+1}} \frac{\sin\theta\, d\theta}{\cos\theta \sin(\theta-\alpha)} = -\beta^2 \int_{\theta_i}^{\theta_{i+1}} [\mathrm{tg}\alpha \sin(\theta-\alpha)^{-1} + \cos^{-1}\alpha \cos^{-1}\theta] d\theta \tag{138}$$

$$= -\beta^2 \{\mathrm{tg}\alpha \log[\sin(\theta-\alpha)(1+\cos(\theta-\alpha))^{-1}] + \cos^{-1}\alpha \log[(1+\sin\theta)\cos^{-1}\theta]\}$$

Combining the expressions Eqs. (137) and (138) one obtains

$$^{i,i+1}\bar{I}_{00}^{0} = \beta^2 \{\mathrm{tg}\alpha \log((1+\cos(\theta-\alpha))\cos^{-1}\theta)$$
$$- \cos^{-1}\alpha \log(1+\sin\theta)\cos^{-1}\theta)\} \Big|_{\theta_i}^{\theta_{i+1}} \tag{139}$$

We return to the original coordinates, but partially retain u_i. Again the definition Eq. (113) are introduced. In addition we set

$$V = -v = X\sin\alpha + Y\cos\alpha \tag{140}$$

$$U = -u = X\cos\alpha - Y\sin\alpha$$

Then (with a different constant of integration)

$$I_{00}^{0}(X,Y,\alpha) = \beta^2 \{\mathrm{tg}\alpha \log(k(R-X)) - \cos^{-1}\alpha \log((R - V)/|U|)\}$$

$$R = (X^2 + Y^2)^{1/2} \tag{141}$$

The basic formulae for the power of the reduced frequency 0 and $m+n \geq 1$ is Eq. (103). We write

$$\bar{I}_{mn}^{0} = \bar{I}_{mn}^{01} + \bar{I}_{mn}^{02}$$

$$\bar{I}_{mn}^{01} = \beta^2(m+n)^{-1} \int(\xi-x)^m(\eta-y)^{n-2}[(\xi-x)d\eta - (\eta-y)d\xi] \qquad (142)$$

$$\bar{I}_{mn}^{02} = -\beta^2(m+n)^{-1} \int(\xi-x)^{m+1}(\eta-y)^{n-2}R^{-1}[(\xi-x)d\eta - (\eta-y)d\xi] \qquad (143)$$

For n = 1, the integrand contains a factor $(\eta-y)^{-1}$. The integrals are then interpreted as principal values.

For n = 0, one has a factor $(\eta-y)^{-2}$. In Eq. (103), (from which Eqs. (142) and (143) arise) no singularity is encountered if $\xi-x > 0$. The singularities in Eqs. (142) and (143) appear because of the separation into terms with and without R. The separation is necessary for the derivation of analytical formulae. In the evaluation of the integrals one follows the procedure described after Eq. (86a).

Introducing sheared coordinates, Eqs. (121) through (123), one obtains

$$^{i,i+1}\bar{I}_{mn}^{01} = \beta^2(m+n)^{-1} \int_{v_i}^{v_{i+1}} (u_{si} + v_s tg\alpha)^m v_s^{n-2} u_{si} dv_s$$

Developing $(u_{si} + v_s tg\alpha)^m$ one obtains elementary integrals. Specializing immediately one obtains

$$^{i,i+1}\bar{I}_{1,0}^{01} = \frac{\beta}{2}{}^2[-(u_{si}^2/v_s) + tg\alpha\, u_{si}\log|kv_s|]\Big|_{v_{si}}^{v_{si+1}} \qquad (144)$$

$$^{i,i+1}\bar{I}_{01}^{01} = \beta^2 u_{si}\log|kv_s|\Big|_{v_{si}}^{v_{si+1}}$$

These expressions will be rewritten in terms of Y and U in Eqs. (147).

The evaluations of ${}^{i,i+1}I^{02}_{mn}$ uses the rotated system of coordinates Eq. (124) through (129).

$$ {}^{i,i+1}\bar{I}^{02}_{mn} = -\beta^2(m+n)^{-1} \int_{v_i}^{v_{i+1}} (u_i\cos\alpha + v\sin\alpha)^{m+1} $$

$$ (-u_i\sin\alpha + v\cos\alpha)^{n-2}(u_i{}^2 + v^2)^{-1/2}u_i dv $$

and with

$$ w = v/u_i $$

$$ {}^{i,i+1}\bar{I}^{02}_{mn} = -\beta^2(m+n)^{-1}u^{m+n} \int_{w_i}^{w_{i+1}} (w\sin\alpha + \cos\alpha)^m $$

$$ (w\cos\ \ - \sin\alpha)^{n-2}(w^2 + 1)^{-1/2}dw $$

For $n \geq 2$ one writes the product of the first two terms in the integrand as a polynomial in w. One then obtains the expressions treated in Appendix C. For $n \leq 2$, one must carry out the division which gives a polynomial (of rather low order in w) and remainder terms with denominators $(w\cos\alpha - \sin\alpha)(w^2 + 1)^{1/2}$ and $(w\cos\alpha - \sin\alpha)^2(w^2 + 1)^{1/2}$. Also these integrals are treated in Appendix C. For the case $m = 1$, $n = 0$, one has

$$ (w\sin\alpha + \cos\alpha)^2(w\cos\alpha - \sin\alpha)^{-2} $$

$$ = tg^2\alpha + 2\sin\alpha\cos^{-2}\alpha(w\cos\alpha - \sin\alpha)^{-1} + \cos^{-2}\alpha(w\cos\alpha - \sin\alpha)^{-2} $$

Then with Eqs. (C.16) and (C.17)

$$^{i,i+1}\bar{I}_{10}^{02} = -\beta^2 \operatorname{sign}(u_i)u_i\{tg^2\alpha \log((w^2 + 1)^{1/2} + w)$$

$$+ \sin\alpha \cos^{-2}\alpha \operatorname{sign}(\cos\theta)\log[\sin(\theta-\alpha)(1+\cos(\theta-\alpha))^{-1}]$$

$$-\cos^{-1}\alpha \operatorname{sign}(\cos\theta)\sin^{-1}(\theta-\alpha)\}$$

The sign functions cancel because of Eq. (133), as can be seen from the following discussion

$$\operatorname{sign}(u_i)\log((w^2 + 1)^{1/2} + w) =$$

$$(\operatorname{sign}(u_i)\log[(u_i{}^2 + v^2)^{1/2} + (\operatorname{sign}u_i)v)/|u_i|]$$

The sign function can obviously be omitted if sign $(u_i) = +1$. Consider now sign $(u_i) = -1$. Then one obtains for the right-hand side

$$-\log[(u_i{}^2 + v^2)^{1/2} - v)/|u_i|] = -\log[u_i{}^2/[|u_i|((u_i{}^2 + v^2)^{1/2} + v)]]$$

$$= \log[((u_i{}^2 + v^2)^{1/2} + v)/|u_i|$$

Therefore,

$$(\operatorname{sign}u_i)\log((w^2 + 1)^{1/2} + w) = \log[(u_i{}^2 + v^2)^{1/2} + v)/|u_i|]$$

One thus obtains (with a change of the constant of integration)

$$^{i,i+1}I_{10}^{02} = \frac{\beta}{2}{}^2u_i\{-tg^2\alpha \log[k((u_i{}^2 + v^2)^{1/2} + v)]$$

$$- \sin\alpha \cos^{-2}\alpha \log[(\eta-\eta_i)/(R + \xi-x)] + \cos^{-1}\alpha R/(\eta-y)\}\Bigg|_{i}^{i+1} \qquad (145)$$

One has for the case $m = 0$, $n = 1$

$$(w\sin\alpha + \cos\alpha)(w\cos\alpha - \sin\alpha)^{-1} = tg\alpha + \cos^{-1}\alpha(w\cos\alpha - \sin\alpha)^{-1}$$

Therefore

$$
{}^{i,i+1}\bar{I}^{02}_{01} = + \beta^2 u_i\{-tg\alpha\log((R+v)|u_i|^{-1}) - \cos^{-1}\alpha \log((\eta-y)[R + (\xi-x)]^{-1}) \Big|_{i}^{i+1} \tag{146}
$$

Collecting these results (Eqs. (144), (145), and (146)) and replacing u_s in $\bar{I}^{1}_{mn}$ by $\cos^{-1}\alpha\ U$ one obtains

$$
\begin{aligned}
I^{01}_{10}(X,Y,\alpha) &= \beta^2\{\cos^{-2}\alpha\ (U^2/Y) - \sin\alpha\ \cos^{-2}\alpha\ U\ \log|kY|\} \\
I^{01}_{01}(X,Y,\alpha) &= -\beta^2\cos^{-1}\alpha\ U\ \log|kY|
\end{aligned} \tag{147}
$$

(In these equations a term log (cos α) has been omitted because it can be incorporated into a constant of integration.)

$$
\begin{aligned}
I^{02}_{10}(X,Y,\alpha) = \beta^2 U\{tg^2\alpha\ \log((R-V)/|U|) - \sin\alpha\ \cos^{-2}\alpha\ \log((R-X)/|Y|) \\
+ \cos^{-1}\alpha\ R/Y\}
\end{aligned} \tag{148}
$$

$$
I^{02}_{01}(X,Y,\alpha) = \beta^2 U\{tg\alpha\ \log((R-V)/|U|) - \cos^{-1}\alpha\ \log((R-X)/|Y|)
$$

See Eqs. (113) and (140) for the definitions of the variables.

For the power 1 of the reduced frequency k, one starts from Eqs. (104). Introducing the rotated system of coordinates Eqs. (124) through (129) one obtains for $\alpha \neq \pi/2$

$$ {}^{i,i+1}\bar{I}^{1}_{mn} = -ik(m+n+1)^{-1} \int_{v_i}^{v_{i+1}} (u_i^2 + v^2)^{-1/2} (u_i\cos\alpha + v\sin\alpha)^m $$

$$ (-u_i\sin\alpha + \cos\alpha)^n u_i dv $$

and with

$$ w = v/u_i $$

$$ {}^{i,i+1}\bar{I}^{1}_{mn} = -ik(m+n+1)^{-1} u_i^{m+n+1} \int_{w_i}^{w_{i+1}} (w^2 + 1)^{1/2} (w\sin\alpha + \cos\alpha)^m $$

$$ (w\cos\alpha - \sin\alpha)^n dw $$

Hence

$$ I^1_{00}(X,Y,\alpha) = +ik\ U \log(R-V)/|U| $$

$$ I^1_{10}(X,Y,\alpha) = +i(k/2)\{\sin\alpha\ UR - \cos\alpha\ U^2 \log((R-V)/|U|) \qquad (149) $$

$$ I^1_{01}(X,Y,\alpha) = +i(k/2)\{\cos\alpha\ UR + \sin\alpha\ U^2 \log[(R-V)/|U|] $$

Eq. (105) is the basic formula for terms of order k^2 and $k^2\log k$. We write as before

$$ \bar{I}^2_{mn} = \bar{I}^{21}_{mn} + \bar{I}^{22}_{mn} + \bar{I}^{23}_{mn} $$

$\bar{I}^{21}_{mn}$ is treated in the rotated coordinate system. The results can be obtained in the "sheared" system of coordinates, but this complicates the limiting process $\alpha\to\pi/2$.

$$ {}^{i,i+1}\bar{I}^{21}_{mn} = (k^2/2)C(m+n+2)^{-1} \int_{v_i}^{v_{i+1}} (\sin\alpha v + \cos\alpha u_i)^m $$

$$ (\cos\alpha v - \sin\alpha u_i)^n u_i dv $$

Hence,

$$ I^{21}_{00}(X,Y,\alpha) = (k^2/4)C\ UV $$

$$ I^{21}_{10}(X,Y,\alpha) = (k^2/6)C\{-\cos\alpha\ U^2V - (1/2)\sin\alpha\ UV^2\} \qquad (150) $$

$$ I^{21}_{01}(X,Y,\alpha) = (k^2/6)C + (\sin\alpha\ U^2V - (1/2)\cos\alpha\ UV^2) $$

I^{22} is treated in the rotated system of coordinates. With $w = v/u_i$ one obtains

$$ {}^{i,i+1}\bar{I}^{22}_{mn} = -\beta^2(k^2/2)(m+n+2)^{-1}u_i^{m+n+2} \int_{w_i}^{w_{i+1}} (w^2 + 1)^{-1/2} $$

$$ (w\sin\alpha + \cos\alpha)^{m+1}(w\cos\alpha - \sin\alpha)^n dw $$

One has

for $m = 0$, $n = 0$

$$ \int(w^2 + 1)^{-1/2}(w\sin\alpha + \cos\alpha)dw = \sin\alpha(w^2 + 1)^{1/2} $$

$$ + \cos\alpha \log[(w^2 + 1)^{1/2} + w], $$

for $m = 1$, $n = 0$

$$ \int(w^2 + 1)^{-1/2}(w\sin\alpha + \cos\alpha)^2 dw = (1/2)\sin^2\alpha w(w^2 + 1)^{1/2} $$

$$ + 2\sin\alpha\cos\alpha(w^2 + 1)^{1/2} + (\cos^2\alpha - (1/2)\sin^2\alpha)\log[(w^2 + 1)^{1/2} + w]\pi $$

and for $m = 0$, $n = 1$

$$\int (w^2 + 1)^{-1/2}(w\sin\alpha + \cos\alpha)(w\cos\alpha - \sin\alpha)dw$$

$$= (1/2)\sin\alpha\ \cos\alpha w(w^2 + 1)^{1/2} + (\cos^2\alpha - \sin^2\alpha)(w^2 + 1)^{1/2}$$

$$- (3/2)\sin\alpha\ \cos\alpha\ \log[(w + 1)^{1/2} + w]$$

Therefore

$$I_{00}^{22} = \beta^2(k^2/4)\{\sin\alpha\ U\ R - \cos\alpha\ U^2\ \log((R-V)/|U|)\}$$

$$I_{10}^{22} = -\beta^2(k^2/6)\{(1/2)\sin^2\alpha\ UVR + 2\sin\alpha\ \cos\alpha\ U^2R$$

$$-(1/2)(3\ \cos^2\alpha - 1)U^3\ \log(R-V)/|U|) \tag{151}$$

$$I_{01}^{22} = -\beta^2(k^2/6)\{1/2\ \sin\alpha\ \cos\alpha\ UVR + (\cos^2\alpha - \sin^2\alpha)U^2R$$

$$+ (3/2)\sin\alpha\ \cos\alpha\ U^3\ \log((R-V)/|U|)$$

Using again the rotated coordinate system one obtains from Eq. (104)

$$^{i,i+1}\bar{I}_{mn}^{23} = (k^2/2)(m+n+2)^{-1} \int_{v_i}^{v_{i+1}} (u_i\cos\alpha + v\sin\alpha)^m(-u_i\sin\alpha + v\cos\alpha)^n$$

$$[\log(k(R + u_i\cos\alpha + v\sin\alpha)) - (m+n+2)^{-1}u_i]dv \tag{152}$$

and with $w = v/u_i$

$$
{}^{i,i+1}\bar{I}^{23}_{mn} = (k^2/2)(m+n+2)^{-1}\ u_i{}^{m+n+2}
$$

$$
\int_{w_i}^{w_{i+1}} (w\sin\alpha + \cos\alpha)^m (w\cos\alpha - \sin\alpha)^n [f(w) - (m+n+2)^{-1}]dw \tag{153}
$$

where

$$
f(w) = \log(ku_i) + \log[(w^2 + 1)^{1/2} + w\sin\alpha + \cos\alpha]
$$

Let

$$
Q_j = \int w^j f(w)dw \tag{154}
$$

Then

$$
{}^{i,i+1}\bar{I}^{23}_{00} = (k^2/4)[u_i{}^2(Q_0 - (1/2)vu_i]_i^{i+1}
$$

$$
\begin{aligned}
{}^{i,i+1}\bar{I}^{23}_{10} = (k^2/6)[&\cos\alpha\ u_i{}^3 Q_0 + \sin\alpha\ u_i{}^3 Q_1 \\
&- (1/3)\cos\alpha\ u_i{}^2 v - (1/6)\sin\alpha\ u_i v^2]_i^{i+1}
\end{aligned} \tag{155}
$$

$$
\begin{aligned}
{}^{i,i+1}\bar{I}^{23}_{01} = (k^2/6)[&-\sin\alpha u_i{}^3\ Q_0 + \cos\alpha\ u_1{}^3 Q_1 \\
&+ (1/3)\sin\alpha\ u_i{}^2 v - (1/6)\cos\alpha\ u_i v^2]_i^{i+1}
\end{aligned}
$$

One has

$$
Q_j = (j+1)^{-1}[w^{j+1}f(w) - \int w^{j+1}f'(w)dw]
$$

Here

$$
f'(w) = [(w^2 + 1)^{1/2} + w\sin\alpha + \cos\alpha]^{-1}[w(w^2 + 1)^{-1/2} + \sin\alpha]
$$

Multiplying numerator and denominator by $(w^2 + 1)^{1/2} - (w\sin\alpha + \cos\alpha)$ one obtains

$$f'(w) = [w\cos\alpha - \sin\alpha]^{-2}[(w^2 + 1)^{1/2} - (w\sin\alpha + \cos\alpha)]$$

$$[w + (w^2 + 1)^{1/2}\sin\alpha](w^2 + 1)^{-1/2}$$

$$f'(w) = [w\cos\alpha - \sin\alpha]^{-1}[\cos\alpha - (w^2 + 1)^{-1/2}]$$

Then, for $j = 0$, after dividing w by $[w\cos\alpha - \sin\alpha]$

$$\int wf'(w)dw = \int[\cos\alpha^{-1} + tg\alpha(w\cos\alpha - \sin\alpha)^{-1}][\cos\alpha - [w^2 + 1)^{-1/2}]dw$$

and with Eq. (C.16)

$$\int wf'(w)dw = w + tg\alpha \log(w\cos\alpha - \sin\alpha) - \cos^{-1}\alpha \log((w^2 + 1)^{1/2} + w)$$

$$-tg\alpha \log[\sin(\theta-\alpha)(1 + \cos(\theta-\alpha))^{-1}]$$

here

$$\theta = \mathrm{arctg} w$$

Moreover, for $j = 1$, again, after carrying out the division

$$\int w^2 f'(w)dw = \int[w\cos^{-1}\alpha + \sin\alpha \cos^{-2}\alpha + tg^2\alpha(w\cos\alpha - \sin\alpha)^{-1}]$$

$$[\cos\alpha - (w^2 + 1)^{-1/2}]dw$$

$$\int w^2 f'wdw = (w^2/2) + tg\alpha\, w + tg^2\alpha \log(w\cos\alpha - \sin\alpha)$$

$$- \cos^{-1}\alpha(w^2 + 1)^{1/2} - \sin\alpha \cos^{-2}\alpha \log((w^2 + 1)^{1/2} + w)$$

$$- tg^2\alpha \log[\sin(\theta-\alpha)(1 + \cos(\theta-\alpha))^{-1}]$$

By substitution of $w = v/u_i$ one obtains

$$f = \log(k|u_i|) + \log[((u_i^2 + v^2)^{1/2} + u_i\cos\alpha + v\sin\alpha)/|u_i|]$$

$$f = \log(k(R + \xi - x))$$

$$\int wf'(w)dw = v/u_i + \mathrm{tg}\alpha \log[k(\eta - y)] - \cos^{-1}\alpha \log[(R+v)/|u_i|]$$
$$- \mathrm{tg}\alpha \log[(\eta - y)(R + \xi - x)^{-1}]$$

$$\int wf'(w)dw = v/u_i - \cos^{-1}\alpha \log((R+v)/|u_i|) + \mathrm{tg}\alpha \log(k(R + (\xi - x))$$

$$\int w^2 f'(w)dw = (1/2)(v/u_i)^2 + \mathrm{tg}\alpha(v/u_i) - \cos^{-1}\alpha(R/u_i)$$
$$- \sin\alpha\cos^{-2}\alpha \log[(R+v)/|u_i|] + \mathrm{tg}^2\alpha \log[k(R + (\xi - x))]$$

$$Q_0 = (V/U - \mathrm{tg}\alpha)\log(k(R-X)) - V/U + \cos^{-1}\alpha \log((R-V)/|U|)$$

$$Q_1 = (1/2)\{(V/U)^2 - \mathrm{tg}^2\alpha)\log(k(R-X)) + \sin\alpha \cos^{-2}\alpha \log((R-V)/|U|)$$
$$- (1/2)(V/U)^2 - \mathrm{tg}\alpha(V/U) - \cos^{-1}\alpha(R/U)\}$$

and rewriting Eqs. (155)

$$I_{00}^{23}(X,Y,\alpha) = (k^2/4)\{U^2 Q_0 - (1/2)VU)\}$$

$$I_{10}^{23}(X,Y,\alpha) = -(k^2/6)\{\cos\alpha\, U^3 Q_0 + \sin\alpha\, U^3 Q_1$$
$$- (1/3)\cos\alpha\, U^2 V - (1/6)\sin\alpha\, UV^2\}$$

$$I_{01}^{23}(X,Y,\alpha) = (k^2/6)\{+\sin\alpha\, U^3 Q_0 - \cos\alpha\, U^3 Q_1$$
$$- (1/3)\sin\alpha\, U^2 V + (1/6)\cos\alpha\, UV^2\}$$

SECTION VI

INTEGRATIONS WITH RESPECT TO X AND Y

With the formulae derived so far, the upwash at a given point x,y can be evaluated in terms of the parameters that describe the pressure distribution. One still has to carry out the integration over the xy-elements. This can be done numerically or analytically. The upwash has singularities as the point x,y approaches the boundaries of the $\xi\eta$-element or of its wake, and special provisions must be made for the terms which give infinities in the upwash. Aside from this one will obtain good results if one uses a sufficiently large number of points xy.

The analytic procedure will be shown in this section. In a numerical approach one will add all (or nearly all) contributions to the upwash before the integrations are carried out; in an analytic procedure one must keep them separate. This gives quite lengthy lists of formulae.

The upwash formulae have arisen from contour integrals around the $\xi\eta$-element. The individual expressions are functions of $X = x-\xi_i$, $Y = y-\eta_i$ and $\alpha_{i,i+1}$; variables U and V are considered as functions of X and Y. The ξ_i's and η_i's are the corners of the $\xi\eta$-element, $\alpha_{i,i+1}$ gives the slope of the side of the $\xi\eta$-element for which the integration has been carried out. The general expressions (for $\alpha \neq \pi/2$) fail if one tries to substitute $\alpha = \pi/2$. Separate formulae for $\alpha = \pi/2$ have therefore been derived. The limiting process $\alpha \to \pi/2$ is shown in Appendix F.

The contribution to the upwash for the form of the pressure distribution assumed here is given by

$$p(x,y)I_{00}(X,Y,\alpha) + c_{10}I_{10}(X,Y,\alpha) + c_{01}I_{01}(X,Y,\alpha),$$

but

$$p(x,y) = p(\xi_i,\eta_i)+(x-\xi_i)c_{10}+(\eta-y_i)c_{01}$$

Thus, one has to evaluate expressions

$$p(\xi_i\eta_i) \iint I_{00}dxdy + c_{10} \iint \tilde{I}_{10}dxdy + c_{01}\iint \tilde{I}_{01}dxdy \tag{156}$$

with

$$\tilde{I}_{10} = I_{10} + XI_{00} \tag{157}$$

$$\tilde{I}_{01} = I_{01} + YI_{00}$$

The integration is carried out over an xy-element. The expressions I consists of a number of summands all of which, except those that contain a factor log (k(R-X)) appear in the form,

$$G(x,y) = R^{\ell}f(\theta) \tag{158}$$

if written in polar coordinates R, θ. Inspecting the expressions I^i_{mn} one finds, that

$$\ell = (i + m + n)$$

The integral over an xy-element then assumes the form

$$\iint G(x,y)dxdy = \iint R^{\ell+1}f(\theta)dRd\theta$$

Here the integration with respect to R can be carried out; the contribution of the lower limit R = 0 vanishes because $\ell > -2$; subsequently, dθ is expressed by $d\theta = (x\,dy - ydx)R^{-2}$. Therefore,

$$\int G(x,y)dxdy = (\ell+2)^{-1} \oint G(x,y)(xdy-ydx) \tag{159}$$

where the integral on the right is taken along the contour of the xy-element.

The expression containing the factor log(k(R-X)) has the form

$$G(x,y) = g(x,y)\log(k(R-X))$$

where

$$g(x,y) = R^{\ell} f(\theta)$$

Therefore,

$$G(x,y) = R^{\ell} f(\theta)[\log(kR) + \log((R-X)/R)]$$

The second term in the bracket depends only upon θ. Therefore,

$$\iint G(x,y)dxdy = \iint R^{\ell+1} f(\theta)[\log((R-X)/R) + \log(kR)]d\theta$$

$$= (\ell+2)^{-1} \oint f(\theta)\{[R^{\ell+2}[\log(R-X)/(R) + \log(kR)]) - (\int_0^{R(\theta)} R^{\ell+1} dR)\}d\theta$$

Hence,

$$\iint G(x,y)dxdy = (\ell+2)^{-1} \oint [G(x,y) - (\ell+2)^{-1} g(x,y)](xdy-ydx) \quad (160)$$

We introduce an angle γ which gives the direction of one side of the xy-element.

$$tg\gamma_{i,i+1} = \frac{x_i - x_{i+1}}{y_i - y_{i+1}} \quad (161)$$

For integrations which are to be carried out along the sides of the xy-element we introduce coordinates p, q:

$$\begin{aligned} X &= p \cos \gamma + q \sin \gamma \; ; \; p = X \cos \gamma - Y \sin \gamma \\ Y &= -p \sin \gamma + q \cos \gamma ; \; q = X \sin \gamma + Y \cos \gamma \end{aligned} \quad (162)$$

Along such a side p is constant; the variable of integration is q. The expressions U and V had been defined by

$$X = U \cos\alpha + V \sin\alpha \; ; \; U = X \cos\alpha - Y \sin\alpha$$
$$Y = -U \sin\alpha + V \cos\alpha \; ; \; V = X \sin\alpha + Y \cos\alpha \tag{163}$$

For the integrations U and V are expressed by p and q

$$U = p \cos\delta + q \sin\delta \; ; \; p = U \cos\delta - V \sin\delta$$
$$V = -p \sin\delta + q \cos\delta \; ; \; q = U \sin\delta + V \cos\delta \tag{164}$$

with

$$\delta = \gamma - \alpha \tag{164a}$$

In the integration along the sides of the xy-elements expressions, xdy-ydx will occur. One finds in general

$$XdY-YdX = pdq-qdp \tag{165}$$

and *since* $p = const$

$$XdY-Ydx = pdq$$

Now we summarize the results of Section V (i.e., the formulae for the functions $I^{\cdot\cdot}_{00}$ and $\tilde{I}^{\cdot\cdot}_{10}$ and $\tilde{I}_{01}$) in a form suitable for the integration. For this purpose we introduce

$$\begin{aligned}
\phi_1 &= \log\,(k(R-X)) = \log\,(R-X)/|Y|) + \log\,(k|Y|) \\
\phi_2 &= \log\,(R-V)/|U|) \\
\phi_3 &= Y^{-1} \\
\phi_4 &= RY^{-1} \\
\phi_5 &= 1 \\
\phi_6 &= R
\end{aligned} \tag{166}$$

These expressions will be regarded as functions of p and q. The expressions $I_{\cdot\cdot}^{\cdot\cdot}$ are linear combinations of these functions with cofficients which are homogeneous in U, V, X and Y. (The explicit dependence upon $\tilde{X}$ and $\tilde{Y}$ occurs because we have to form $\tilde{I}_{10} = I_{10} + XI_{00}$ and $\tilde{I}_{01} = I_{01} + XI_{00}$.) These coefficients are considered as scalar products, written as products of row matrices and column matrices. The row matrix depends only upon α, the column matrices are homogeneous functions in (U,V,X,Y). In the following expressions, only those terms actually encountered in the expressions I are included. We introduce

$$
\begin{aligned}
[w_1] &= [U,X,Y]^+ \\
[w_2] &= [U^2,UV,UX,UY]^+ \qquad (167) \\
[w_3] &= [U^3,U^2V,UV^2,U^2X,UVX,U^2Y,UVY]^+
\end{aligned}
$$

The row matrices belonging to a function I_{mn}^{ℓ} (or $\tilde{I}_{mn}^{\ell}$) are denoted by c with the same indices as the function I and a third subscript referring to the function ϕ to which they belong. Notice that the coefficients occurring in a specific function I may be of different degree, because the functions ϕ_j have different degrees in U and V. Only some of the function ϕ will occur in a given expression I.

For I_{00}^{0} the matrix $[w_0]$ and the vectors $C_{00}^{0}, \ldots$ are scalars. These expressions are written down directly. One then obtains the following list.

$$
\begin{aligned}
I_{00}^{0} &= \beta^2 tg\alpha\phi_1 - \cos^{-1}\alpha\phi_2 \\
\tilde{I}_{10}^{0} &= \phi_1 + [c_{10,1}^{0}][w_1]\phi_1 + [c_{10,2}^{0}][w_1]\phi_2 \\
&\quad + [c_{10,3}][w_2]\phi_3 + [c_{10,4}^{0}][w_1]\phi_4 \qquad (168) \\
\tilde{I}_{01}^{0} &= [c_{01,1}^{0}][w_1]\phi_1 + [c_{01,2}^{0}][w_3]\phi_2
\end{aligned}
$$

$$I^1_{00} = [c^1_{00,2}][w_1]\phi_2$$

$$\tilde{I}^1_{mn} = [c^1_{mn,2}][w_2]\phi_2 + [c^1_{mn,6}][w_1\phi_6] \qquad , m+n = 1 \qquad \begin{matrix}(168)\\ (\text{cont'd})\end{matrix}$$

$$I^2_{00} = [c^2_{00,1}][w_2]\phi_1 + [c^2_{00,2}][w_2]\phi_2 + [c^2_{00,5}][w_2]\phi_5 + [c^2_{00,6}][w_1]\phi_6$$

$$\tilde{I}^2_{mn} = [c^2_{mn,1}][w_3]\phi_1$$

$$+ [c^2_{mn,2}][w_3]\phi_2 + [c^2_{mn,5}][w_3]\phi_5 + [c^2_{mn,6}][w_2]\phi_6 \; ; \; m+n = 1$$

The vectors [c], ordered according to their dimensions (which, of course, matches the dimension of the vectors [w]), are given by

$$[c^0_{10,1}] = [-\beta^2 \sin\alpha \cos^{-2}\alpha,\ \beta^2 \mathrm{tg}\alpha,\ 0]$$

$$[c^0_{10,2}] = [\beta^2 \mathrm{tg}^2\alpha,\ -\beta^2\cos^{-1}\alpha,\ 0]$$

$$[c^0_{10,4}] = [\beta^2\cos^{-1}\alpha,\ 0,\ 0]$$

$$[c^0_{01,1}] = [-\beta^2\cos^{-1}\alpha,\ 0,\ \beta^2\mathrm{tg}\alpha]$$

$$[c^0_{01,2}] = [\beta^2\mathrm{tg}\alpha,\ 0,\ -\beta^2\cos^{-1}\alpha]$$

$$[c^1_{00,2}] = [ik,\ 0,\ 0] \tag{169}$$

$$[c^1_{10,6}] = [(ik/2)\sin\alpha,\ 0,\ 0]$$

$$[c^1_{01,6}] = [(ik/2)\cos\alpha,\ 0,\ 0]$$

$$[c^2_{00,6}] = [(k^2/4)\beta^{-2}\sin\alpha,\ 0,\ 0]$$

$$[c^0_{10,3}] = [\beta^2\cos^{-2}\alpha,\ 0,\ 0,\ 0]$$

$$[c^1_{10,2}] = [-(ik/2)\cos\alpha,\ 0,\ ik,\ 0]$$

$$[c^1_{01,2}] = [(ik/2)\sin\alpha,\ 0,\ 0,\ ik]$$

$$[c^2_{00,1}] = [-(k^2/4)tg\alpha,\ (k^2/4),\ 0,\ 0]$$

$$[c^2_{00,2}] = (k^2/4)(\cos^{-1}\alpha-\beta^{-2}\cos\alpha),\ 0,\ 0,\ 0]$$

$$[c^2_{00,5}] = [0,\ (k^2/4)(C-3/2),\ 0,\ 0]$$

$$[c^2_{10,6}] = [(k^2/6)((1/2)tg\alpha-2\beta^{-2}\sin\alpha\cos\alpha),\ (k^2/12)\beta^{-2}\sin\alpha,\ (k^2/4/\beta^{-2}\sin\alpha,0]$$

$$[c^2_{01,6}] = [(k^2/6)((1/2)-\beta^{-2}(\cos^2\alpha-\sin^2\alpha),-(k^2/2)\beta^{-2}\sin\alpha\cos\alpha,\ 0,\ (k^2/4)\beta^{-2}\sin\alpha]$$

$$[c^2_{10,1}] = [(k^2/6)\sin\alpha\ (1+(1/2)tg^2\alpha),-(k^2/6)\cos\alpha,\ -(k^2/12)\sin\alpha,\ -(k^2/4)tg\alpha,\ k^2/4,\ 0,\ 0] \tag{169}$$

(cont'd)

$$[c^2_{10,2}] = [(k^2/6)(-(1+(1/2)tg^2\alpha)+(1/2)\beta^{-2}(3\cos^2\alpha-1),0,0,\ (k^2/4)(\cos^{-1}\alpha-\beta^{-2}\cos\alpha),\ 0,\ 0,\ 0]$$

$$[c^2_{10,5}] = [0,(k^2/6)((4/3)-C)\cos\alpha+(1/2)\sin\alpha tg\alpha),\ (k^2/6)((5/12)-(C/2))\sin\alpha,\ 0,\ (k^2/4)(C-3/2),\ 0,\ 0]$$

$$[c^2_{01,1}] = [-(k^2/12)\sin\alpha tg\alpha,\ (k^2/6)\sin\alpha,\ -(k^2/12)\cos\alpha,\ 0,\ 0,\ -(k^2/4)tg\ \alpha,\ (k^2/4)]$$

$$[c^2_{01,2}] = [-(k^2/12)(tg\alpha-3\beta^{-2}\sin\alpha\cos\alpha,0,\ 0,\ 0,\ 0,\ -(k^2/4)tg\ \alpha,\ (k^2/4)]$$

$$[c^2_{01,5}] = [0,(k^2/6)(C-5/6)\sin\alpha,\ (k^2/6)((5/12)-(C/2)\cos\alpha,\ 0,\ 0,\ 0,\ (k^2/4)C-3/2)]$$

The factors $tg\alpha$ and $\cos^{-1}\alpha$ which occur in some of the coefficients are indications that some of these formulae are not applicable for $\alpha = \pi/2$. The results necessary for $\alpha = \pi/2$ have been obtained by direct computation in Section V. They are brought into a form analogous to that for $\alpha \neq \pi/2$. The functions ϕ remains the same. The function ϕ_2 does not appear. The vectors $[w_j]$ assume different forms

$$[\tilde{w}_1] = [X,Y]^+,\ [\tilde{w}_2]^+ = [X^2,XY,Y^2]^+,\ [\tilde{w}_3] = [X^2Y,XY^2,Y^3]^+ \tag{170}$$

Then

$$\begin{aligned}
I_{00}^{0}(\pi/2) &= [\beta^2,0][\tilde{w}_1]\phi_3 + \beta^2\phi_4 \\
\tilde{I}_{10}(\pi/2) &= [0,\beta^2/2][\tilde{w}_1]\phi_1 + [3\beta^2/2,0,0][\tilde{w}_2]\phi_3 + [3\beta^2/2,0][\tilde{w}_1]\phi_4 \\
\tilde{I}_{01}^{0}(\pi/2) &= [2\beta^2,0][\tilde{w}_1]\phi_5 + 2\beta^2\phi_6 \\
I_{00}^{1}(\pi/2) &= [0,\ -ik][\tilde{w}_1]\phi_1 \\
\tilde{I}_{10}^{1}(\pi/2) &= [0,\ -ik,\ 0][\tilde{w}_2]\phi_1 + [0,-ik/2][\tilde{w}_1]\phi_6 \\
\tilde{I}_{01}^{1}(\pi/2) &= [0,0,\ -ik/2][\tilde{w}_2]\phi_1 \\
I_{00}^{2}(\pi/2) &= [0,-k^2/4,0][\tilde{w}_2]\phi_1 + [0,(k^2/4)(1-C)][\tilde{w}_2]\phi_5 \\
&\quad + [0,(k^2/4)(1-\beta^{-2})][\tilde{w}_1]\phi_6 \\
I_{10}^{2}(\pi/2) &= [-k^2/6,0,(k^2/12)(\beta^{-2} \\
&\quad + (1/2))][\tilde{w}_3]\phi_1 + [7k^2/72,-C/6,0,0][\tilde{w}^3]\phi_5 \\
&\quad + [0,k^2(-(\beta^{-2}/6)-(5/24),0][\tilde{w}_2]\phi_6 \\
I_{01}^{2}(\pi/2) &= [0,-k^2/12,0][\tilde{w}_3]\phi_1 + [0,0,k^2(-(C/12) + (5/18))][\tilde{w}_3]\phi_5 \\
&\quad + [0,0,-(\beta^{-2}/12) - (1/12)][\tilde{w}_2]\phi_6
\end{aligned} \tag{171}$$

So far this amounts only to repetitions in a different form of the formulae of Section V. To carry out the integration along the side of the xy-elements for the case $\alpha = \pi/2$ one must express the functions U,V,X, and Y, occurring in the vectors $[w_j]$, by p, and q. Along the path of integration p is constant and q is the variable of integration.

Therefore, one writes for $\alpha \neq \pi/2$

$$[w_1] = M_1[p,q]^+$$

$$[w_2] = M_2[p^2,pq,q^2]^+$$

$$[w_3] = M_3[p^3,p^2q,q^3]^+$$

where M_1, and M_2 are respectively 3 by 2, 4 by 3 and 7 by 4 matrices. The rows of the matrices are the coefficients obtained by expressing, one element of the column matrix on the left in terms of p and q. (One has for instance in $[w_2]$ $U^2 = \cos^2\delta p^2 + 2\cos\delta\sin\delta pq + \sin^2\delta q^2$) and one will indeed find in M_2 as first row: $\cos^2\delta$, $2\cos\delta\sin\delta$, $\sin^2\delta$. One obtains the following expressions.

$$M_1 = \begin{bmatrix} \cos\delta & \sin\delta \\ \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \tag{172}$$

$$M_2 = \begin{bmatrix} \cos^2\delta & 2\sin\delta\cos\delta & \sin^2\delta \\ -\sin\delta\cos\delta & \cos^2\delta-\sin^2\delta & \sin\delta\cos\delta \\ \cos\delta\cos\gamma & \sin\delta\cos\gamma+\cos\delta\sin\gamma & \sin\delta\sin\gamma \\ -\sin\delta\cos\gamma & \cos\delta\cos\gamma-\sin\delta\sin\gamma & \sin\delta\cos\gamma \end{bmatrix} \tag{173}$$

$$M_3 = \begin{bmatrix} \cos^3\delta & 3\sin\delta\cos^2\delta & 3\cos\delta\sin^2\delta & \sin^3\delta \\ -\sin\delta\cos^2\delta & -2\sin^2\delta\cos\delta+\cos^3\delta & -\sin^3\delta+2\sin\delta\cos^2\delta & \sin^2\delta\cos\delta \\ \sin^2\delta\cos\delta & -2\sin\delta\cos^2\delta+\sin^3\delta & \cos^3\delta-2\sin^2\delta\cos\delta & \sin\delta\cos^2\delta \\ \cos^2\delta\cos\gamma & 2\sin\delta\cos\delta\cos\gamma+\cos^2\delta\sin\gamma & \sin^2\delta\cos\gamma+2\sin\delta\cos\delta\sin\gamma & \sin^2\delta\sin\gamma \\ -\sin\delta\cos\delta\cos\gamma & (\cos^2\delta-\sin^2\delta)\cos\gamma-\sin\delta\cos\delta\sin\gamma & \sin\delta\cos\delta\cos\gamma+(\cos^2\delta-\sin^2\delta)\sin\gamma & \sin\delta\cos\delta\sin\gamma \\ -\cos^2\delta\sin\gamma & -2\sin\delta\cos\delta\sin\gamma+\cos^2\delta\cos\gamma & -\sin^2\delta\sin\gamma+2\sin\delta\cos\delta\cos\gamma & \sin^2\delta\cos\gamma \\ \sin\delta\cos\delta\sin\gamma & -(\cos^2\delta-\sin^2\gamma)\sin\gamma-\sin\delta\cos\delta\cos\gamma & -\sin\delta\cos\delta\sin\gamma+(\cos^2\delta-\sin^2\delta)\cos\gamma & \sin\delta\cos\delta\cos\gamma \end{bmatrix} \quad (174)$$

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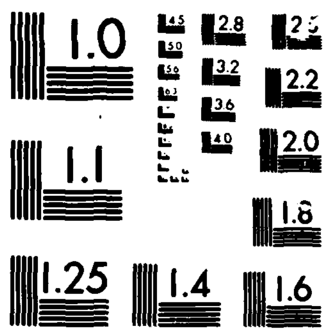

MICROCOP' CHART

For $\alpha = \pi/2$ one sets

$$\tilde{w}_1 = [\tilde{M}_1][p,q]^+$$

$$\tilde{w}_2 = [\tilde{M}_2][p^2,pq,q^2]^+$$

$$\tilde{w}_3 = [\tilde{M}_3][p^2,pq,q^2]^+$$

The rows of $\tilde{M}_1$, $\tilde{M}_2$, and $\tilde{M}_3$ are the coefficients of the development of X, Y, X^2, XY etc. in terms of p and q.

$$\tilde{M}_1 = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix} \tag{175}$$

$$\tilde{M}_2 = \begin{bmatrix} \cos^2\gamma & 2\sin\gamma\cos\gamma & \sin^2\gamma \\ -\sin\gamma\cos\gamma & \cos^2\gamma-\sin^2\gamma & \sin\gamma\cos\gamma \\ \sin^2\gamma & -2\sin\gamma\cos\gamma & \cos^2\gamma \end{bmatrix}$$

(175)
(cont'd)

$$\tilde{M}_3 = \begin{bmatrix} \cos^3\gamma & 3\cos^2\gamma\sin\gamma & 3\sin\gamma\cos^2\gamma & \sin^3\gamma \\ -\sin\gamma\cos^2\gamma & \cos^3\gamma-2\sin^2\gamma\cos\gamma & -\sin^3\gamma+2\sin\gamma\cos^2\gamma & \sin^2\gamma\cos\gamma \\ \sin^2\gamma\cos\gamma & \sin^3\gamma-2\sin\gamma\cos^2\gamma & \cos^3\gamma-2\sin^2\gamma\cos\gamma & \sin\gamma\cos^2\gamma \\ -\sin^3\gamma & 3\sin^2\gamma\cos\gamma & -3\sin\gamma\cos^2\gamma & \cos^3\gamma \end{bmatrix}$$

Now consider some term of one function I^i_{mn} . Take for instance a term containing the column matrix $[w_2]$. Then one has

$$[c_{\ldots,j}][w_2]\phi_j = \{[c_{\ldots,j}][M_2]\}\{[p^2,pq,q^2]^+\phi_j\}$$

The expression within the first brace on the right is a row vector which depends solely on α and γ. The integration along a side of the xy-element operates only on the elements of the vector

$$[p^2\phi_j,pq\phi_j,q^2\phi_j]^+ = \begin{bmatrix} p^2 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_j \\ q\phi_j \\ q^2\phi_j \end{bmatrix}$$

Applying Eq. (129) and observing that xdy-ydx = pdq one obtains as a contribution to $\tilde{I}^i_{mn}$

$$(i+m+n)^{-1} \begin{bmatrix} p^3 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p \end{bmatrix} \begin{bmatrix} \int \phi_j dq \\ \int q_j\phi_j dq \\ \int q^2\phi_j dq \end{bmatrix} \tag{176}$$

For ϕ_1 = log(k(R-X) one has to apply Eq. (130); the function g(xy) is given by 1, q, and q^2. One obtains

$$(i+m+n)^{-1} \begin{bmatrix} p^3 & 0 & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & p \end{bmatrix} \begin{bmatrix} \int \phi_1 dq \\ \int q_2\phi_1 dq \\ \int q^2\phi_1 dq \end{bmatrix} - (i+m+n)^{-1} \begin{bmatrix} \int dq \\ \int q dq \\ \int q^2 dq \end{bmatrix} \tag{177}$$

We denote by $\psi_j^{(\ell)}$ the vector $[\phi_j dq,\ q\phi_j dq,\ q^2\phi_j dq \ldots]^+$ truncated after the ℓ^{th} term.

The contribution to the integral over the xy-element of one endpoint of one side of the xy-element with angle γ is denoted by $J^{\cdot\cdot}_{\cdot\cdot}(\alpha,\gamma)$, with the same indices as the functions $I^{\cdot\cdot}_{\cdot\cdot}$ from which it originates. Ultimately, $J^{\cdot\cdot}_{\cdot\cdot}$ depends on $X = x_j - \xi_i$ and $Y = y_j - \eta_i$ (besides α and γ). Then one has

$$J^{\ell}_{mnj}(\alpha,\gamma) = (\ell+m+n+2)^{-1}\{[c^{\ell}_{mn,j}][w]\begin{bmatrix} \cdot & & \\ & \cdot\ p^2 & \\ & & p \end{bmatrix}[\psi_j],\ j \neq 2 \qquad (178)$$

$J^{\ell}_{m,n}$ is then the sum of all pertinent expressions $J^{\ell}_{m,nj}$. (See Eqs. (168) and (171).) The dimensions of the row matrix $[w]$, the diagonal matrix containing the powers of p and of the vector $[\psi_j]$ have not been shown. The subscript of w is found in Eq. (168). The number of rows in w gives the dimension of the following square matrix and of the vector $[\psi]$. For j = 1 one has

$$J^{\ell}_{mn,1} = [\ell+m+n+2]^{-1}[c^{\ell}_{mn,1}][w_{m+n+\ell}]\begin{bmatrix} p^{m+n+\ell} & & \\ & p^{m+n+\ell-1} & \\ & \cdot & \\ & & p \end{bmatrix}[\psi_1{}^{m+n+\ell}]$$

$$- (\ell+m+n+2)^{-1}[\psi_5^{m+n+\ell}] \qquad (179)$$

As preparation for the evaluation of the vectors $[\psi]$, we first list some recurring auxiliary expressions. In the computation they, and also the vector $[\psi]$, can be evaluated immediately as definite integrals between two limits for q.

$$\int R^{-1}dq = -\log(k(R-q))$$

$$\int R^{-1}qdq = R$$

$$\int R^{-1}q^2dq = \frac{1}{2}((Rq+p^2 \log(k(R-q))$$

$$\int R^{-1}q^3dq = R[(1/3)q^2 - (2/3)p^2]$$

$$\int R^{-1}q^4dq = R[(1/4)q^3 - (3/8)p^2q] - (3/8)p^4 \log(k(R-q) \tag{180}$$

$$\int Rdq = \frac{1}{2}(Rq - p^2 \log(k(R-V))$$

$$\int Rqdq = 1/3\ R^3$$

$$\int Rq^2dq = R[(1/4)q^3 + (1/8)p^2q] + 1/8\ p^4 \log(k(R-V))$$

$$\int \frac{dq}{RY} = \int \frac{dq}{R(q\cos\gamma - p\sin\gamma)} = \frac{1}{p}\log(Y/(R+X)) = +\frac{1}{p}\log(R-X)/Y)$$

The last relation follows from Eq. (E.2).

The general formula for $\int q^m \phi_1 dq$ is Eq. (E.3). Here we specialize

$$\int \phi_1 dq = \{(q - p\operatorname{tg}\gamma)\log(k(R-X)) - p\cos^{-1}\gamma \int R^{-1}dq - q\}$$

$$\int \phi_1 qdq = \frac{1}{2}\{(q^2 - (p\operatorname{tg}\gamma^2)\log(k(R-X)) - p\cos^{-1}\gamma \int R^{-1}(q+(p\operatorname{tg}\gamma))dq$$
$$-(q^2/2) - (p\operatorname{tg}\gamma)q\}$$

$$\begin{aligned}\int \phi_1 q^2dq = (1/3)\{&(q^3 - (p\operatorname{tg}\gamma)^3\log(k(R-X)) \\ &- p\cos^{-1}\gamma \int R^{-1}(q^2 + (p\operatorname{tq}\gamma)q + (p\operatorname{tg}\gamma)^2)dq \\ &- (q^3/3) - (p\operatorname{tg}\gamma)(q^2/2) - (p\operatorname{tg}\gamma)^2\}\end{aligned} \tag{181}$$

$$\begin{aligned}\int \phi_1 q^3dq = (1/4)\{&(q^4 - (p\operatorname{tg}\gamma))^4\log(k(R-X)) \\ &- p\cos^{-1}\gamma \int R^{-1}(q^3 + (p\operatorname{tg}\gamma)q^2 + (p\operatorname{tg}\gamma)^2q + (p\operatorname{tg}\gamma)^3)dq \\ &- (q^4/4) - p\operatorname{tg}\gamma(q^3/3) - (p\operatorname{tg}\gamma)^2(q^2/2) - (p\operatorname{tg}\gamma)^3q\}\end{aligned}$$

From the factor of log $(k(R-X))$, that is from $q^n-(ptg\gamma)^n$, one can split off a factor $(q-ptg\gamma) = Y\cos^{-1}\gamma$. Furthermore from Eqs. (E.4)

$$\int \phi_2 dq = \{(q - (-p\cot\delta)\log((R-V)/U) + p\sin^{-1}\delta \int R^{-1}dq\}$$

$$\int \phi_2 q dq = (1/2)\{(q^2 - (-p\cot\delta)^2)\log((R-V)/U) + p\sin^{-1}\delta\, R^{-1}(q + (-p\cot\delta)dq\} \tag{182}$$

$$\int \phi_2 q^2 dq = (1/3)\{(q^3 - (-p\cot\delta)^3)\log((R-V)/U) + p\sin^{-1}\delta \int R^{-1}(q^2 + (-p\cot\delta)q + (-p\cot\delta)^2 dq\}$$

$$\int \phi_2 q^3 dq = (1/4)\{(q^4 - (-p\cot\delta)^4)\log(R-V)/U) + p\sin^{-1}\delta \int R^{-1}(q^3 + (-p\cot\delta)q^2 + (-p\cot\delta)^2 q + (-p\cot\delta)^3)dq\}$$

From $q^n - (-p\cot\delta)^n$ a factor $q + p\cos\delta = U\sin^{-1}\delta$ can be split off.

$$\int \phi_3 dq = \cos^{-1}\gamma\log(kY)$$

$$\int \phi_3 q dq = \cos^{-1}\gamma[q+(ptg\gamma)\log(kY)] \tag{183}$$

$$\int \phi_3 q^2 dq = \cos^{-1}\gamma[(q^2/2)+(ptg\gamma)q+(ptg\gamma)^2\log(kY)]$$

$$\int \phi_3 q^3 dq = \cos^{-1}\gamma[(q/3)+(ptg\gamma)(q^2/2)+(ptg\gamma)^2 q+(ptg\gamma)^3\log(k(Y))]$$

$$\int \phi_4 dq = \int RY^{-1}dq = p^2\cos^{-2}\gamma \int R^{-1}Y^{-1}dq+\cos^{-1}\gamma[\int \frac{q}{R}\,dq+(ptg\gamma)\int \frac{dq}{R}]$$

$$\int \phi_4 q dq = p^3 tg\gamma\cos^{-2}\int R^{-1}Y^{-1}dq+\cos^{-1}\gamma \int(R^{-1}q^2+(ptg\gamma)R^{-1}q)dq + p^2\cos^{-3}\gamma \int R^{-1}dq$$

$$\int \phi_5 q^n dq = \frac{1}{n+1}\, q^{n+1} \quad (n = 0, 1, 2, 3)$$

$$\int \phi_6 q^n dq = \int q^n R dq, \; n = 0, 1, 2$$

One obviously needs special formulae in those integrals where $\cos^{-1}\alpha$ and $tg\alpha$ or $\sin^{-1}\delta$ and $\cos\delta$ occur, if respectively $\alpha \to \pi/2$ or $\delta \to 0$. They are found by direct integration. For $\gamma = \pi/2$ one has $q = X$, $p = -Y$. Then

$$\int \phi_1 q^n dq = \int X^n \log(k(R-X))dx = \frac{1}{n+1}\left[X^{n+1} \log(k(R-X)) + \int X^{n+1}R^{-1}dX\right]$$

The integrals are immediately found from Eqs. (180) by replacing q by X, and p by -y. Therefore, for $\gamma = \pi/2$

$$\int \phi_1 dq \quad = X \log(k(R-X) + R$$

$$\int \phi_1 q dq \quad = 1/2\{[\log(k(R-X)) + \frac{1}{2}(RX + Y^2\log(k(R-X))\}$$

$$\int \phi_1 q^2 dq = (1/3)\{[X^3\log(k(R-X)) + (1/3)R(X^2-2Y^2)\} \tag{184}$$

$$\int \phi_1 q^3 dq = 1/4\{X^4\log(k(R-X))+R(1/4)X^3-(3/8)XY^2)-(3/8)Y^4\log(k(R-X))\}$$

$$\int \phi_3 q^n dq = Y^{-1} \int X^n dx = Y^{-1}(n+1)^{-1}X^{n+1} \quad , \quad n = 0, 1, 2, 3.$$

$$\int \phi_4 q^n dy = Y^{-1} \int X^n R dx$$

$$\int \phi_4 dy = Y^{-1}[(RX-Y^2\log(k(R-X))]$$

$$\int \phi_4 q dy = Y^{-1}(1/3)R$$

The limit $\delta = 0$ requires a revision of the integrals involving ϕ_2. Then $p = U$, $q = V$, and one obtains

$$\int q^n \log(R-V/|U|)dq = \int V^n \log((R-V)/U)dv$$

$$= (n+1)^{-1}[V^{n+1}\log(R-V)/U) + \int V^{n+1}R^{-1}dV$$

$$\int \phi_2 dq \quad = \{V\log(R-V/U) + R \tag{185}$$

$$\int q\phi_2 dq \quad = (1/2)\{V^2\log((R-V)/U) + 1/2((RV + U^2\log((R-V)/Y)\}$$

$$\int q^2 \phi_2 dq = (1/3)\{V^3 \log(R-V)/U) + R((1/3)V^2 - (2/3)U^2)\} \quad (185)$$

(cont'd)

$$\int q^3 \phi_2 dq = (1/4)\{V^4 \log((R-V)/U) + R((1/4)V^3 - 3/8V^2U - (3/8)U^4 \lg((R-V)/U)\}$$

SECTION VII
SINGULARITIES OF THE UPWASH FIELD

In carrying out the integrations over X and Y one must be aware of the singularities in the upwash field and how different terms contribute to them. Consider a single $\xi\eta$-element. Singularities will probably occur at its boundaries and at the boundaries of the wake and at the wake of points of the contour of the $\xi\eta$-element for which the contour is discontinuous. (see Figs. 3). The wake has already been studied in Section III, but one must ask how these wake singularities express themselves by the formulae found in Section V.

Because of the denominator Y the strongest singularities are caused by the functions ϕ_3 and ϕ_4. Assume first that $\Delta\bar{p}$ is constant throughout the element, then only I_{00}^{0} will contain ϕ_3 and ϕ_4. One has as the only contribution

$$I_{00}^{0}(\pi/2) = \beta^2 Y^{-1}(X+R) = \beta^2(Y-\eta_i)^{-1}[(x-\xi_i)+((x-\xi_i)^2+(y-\eta_i)^2)^{1/2}]$$

For the triangular element, shown in Fig. 3, this must be evaluated for the side 2,3. To recognize the analytic behavior, we rewrite this expression. One has for $x < \xi_3 < \xi_2$

$$Y^{-1}(X+R) = Y/(R-X) = Y/2R$$

Here the flow field does not have a singularity. One has for

$$x > \xi_2 > \xi_3$$

$$Y^{-1}(X+R) = Y^{-1}(2X + Y(R-X)) = 2XY^{-1} \text{ for Y small}$$

This displays the singularity caused by one of the limits ξ_i η_i. We know from Section III, that except for a factor $\exp(ikx)$, the singularity is independent of x. This is seen if one substitutes the limits (ξ_2 and ξ_3), i.e. replaces X by $x-\xi_2$ and $x-\xi_3$. One then obtains

$$2Y^{-1}[(x-\xi_3) - (x-\xi_3)] = 2Y^{-1}(\xi_2-\xi_3)$$

The situation is more complicated if $\Delta\bar{p}$ is linear in ξ and η. The contribution of ϕ_3 and ϕ_4 then appear

in I^0_{00} $(\pi/2)$, namely, $\beta^2 Y^{-1}(X+R)$

in I^0_{10} $(\pi/2)$, namely, $(\beta^2/2)XY^{-1}(X+R)$

and in I^0_{10} (π), namely, $\beta^2Y^{-1}(\cos^{-2}\alpha U^2 + \cos^{-1}\alpha UR)$

The last term is rewritten for the vicinity of $Y = 0$. There $U = X \cos\alpha$. Therefore, except for terms of higher order in Y

$$\beta^2Y^{-1}(\cos^{-2}\alpha U^2 + \cos^{-1}\alpha UR) = \beta^2Y^{-1}X(X+R)$$

As before, no singularities will be encountered for $x < \xi_2 < \xi_3$. In the following discussion for $x > \xi_2 > \xi_3$ we retain only the singular parts.

$$\beta^2Y^{-1}(X+R)+ 2\beta^2Y^{-1}X$$

$$\beta^2/2Y^{-1}X(X+R)+ \beta^2Y^{-1}X^2$$

$$\beta^2Y^{-1}(\cos^{-2}\alpha U^2 + \cos^{-1}\alpha UR)+ 2\beta^2Y^{-1}X$$

Now choose a function $\Delta\bar{p}(\xi,\eta)$, whose integral, taken for $\eta = \eta_2 = \eta_3$ from ξ_2 to ξ_3, is zero. Such a function should not generate a wake singularity:

$$\Delta\bar{p}(\xi,\eta_2) = c_{10}(\xi-(\xi_2+\xi_3)/2)$$

In order to evaluate the upwash, one must first evaluate

$$\Delta\bar{p}(x,y) = \Delta p(x,\eta_2) = c_{10}(x-(\xi_2+\xi_3)/2)$$

Then one forms

$$\Delta\bar{p}(x,y)I_{00}^0 + c_{10}I_{10}^0$$

The integrals must be formed around the entire $\xi\eta$-element. Only the limits due to the points 2 and 3 will contribute to the singularity. Point 2 is the upper limit for the integration along the side 1,2 and the lower limit for the side 2,3. Point 3 is the upper limit for the integration along 2,3 and the lower limit for the integration along 3,1. One obtains the following contributions

from $I_{00}^0(\pi/2)$ $\quad \beta^2 Y^{-1} c_{10} 2[x-(\xi_2+\xi_3)/2]((x-\xi_3)-(x-\xi_2))$

from $I_{10}^0(\pi/2)$ $\quad \beta^2 Y^{-1} c_{10}[(x-\xi_3)^2 - (x-\xi_2)^2]$

from $I_{00}^0(\alpha_{12})$ $\quad \beta^2 Y^{-1} c_{10} 2(x-\xi_2)^2$

from $I_{10}^0(\alpha_{31})$ $\quad -\beta^2 Y^{-1} c_{10} 2(x-\xi_3)^2$

The sum of these expressions cancels, indeed.

In the analytic integration over x and y, ϕ_3 and ϕ_4 give separate formulae. The Y^{-1} singularity (which appears in the individual expression) becomes a log Y singularity after the integration. If $Y = 0$ is one of the limits of integration, then one obtains infinity. It is true that these infinities will always cancel. This happens for points upstream of the $\xi\eta$-element because no singularities are present, for wake points if $\Delta\bar{p}$ is continuous because of the contribution of an adjacent $\xi\eta$-element; for wake points if $\Delta\bar{p}$ is discontinuous because the region of integration will then extend across the wake and the Y^{-1} singularity changes sign as one crosses the wake. In any case the formulae are needed for the numerical procedure which show the singularity separately. They will be derived later.

For elements where $\alpha \neq \pi/2$ for all sides, no Y^{-1} singularities will appear. As mentioned above, this is of interest if one works with elements in which $\Delta\bar{p}$ is constant for then one can choose the individual elements as pressure and upwash areas.

Weaker singularities occur in ϕ_1 and ϕ_3. In $\phi_1 = \log(k(R-X))$ the argument of logarithm becomes zero for $Y = 0$ and $X > 0$. To display the singularity one writes

$$\phi_1 = \log(k(R-X) = \log kY^2(R+X)^{-1} = \log(kY^2) - \log(R+X)$$

In some of the expressions ϕ_1 is multiplied by powers of Y, which makes the singularity less pronounced. Upon integration with respect to x and y one obtains finite quantities. One does not depend upon cancellation of different terms, but one must make sure, that the formulae for the integrals do not, inadvertantly, give the difference beween two infinite quantities.

For $I^{0}_{..}(\pi/2)$, the expression ϕ_1 does not introduce singularities except at the boundary of the $\xi\eta$ element. This is obvious for points $Y = 0$ if $x < \xi_3 < \xi_3$. For $x > \xi_2 > \xi_3$ one obtains after substitution of the limits

$$[Y^n\log(kY^2) - Y^n\log(2(x-\xi_3))] - [Y^n\log(kY^2) - \log(2(x-\xi_2))]$$

The logarithmic terms cancel.

However, the same argument cannot be made for $\xi_3 < x < \xi_2$. The expression due to ξ_2 does not generate a singularity, the singularity of the expression due to ξ_3 remains uncompensated.

In $I^{23}_{00}(\pi/2)$, the function ϕ_1 is multiplied by X, X^2 etc. Then the singularity will not cancel for $x > \xi_2 > \xi_3$; one has a logarithmic contribution to the wake singularity. This is to be expected if one examines the portion K_1 of the kernel K; $I^{23}_{..}$ arises from the lowest order term in k that contains the logarithm.

In the expressions for $\alpha \neq \pi/2$, the function ϕ_1 is encountered in the following forms

in I^{0}_{00} (α) $\qquad \beta^2 \mathrm{tg}\alpha\phi_1$

in I^{02}_{10} (α) $\qquad -\beta^2 \sin\alpha \cos^{-2}\alpha U\phi_1$,

and in I^{02}_{01} (α) $\qquad -\beta^2 \cos^{-1}\alpha U\phi_1$

Only an integration limit for which Y = 0 will contribute to the singular part. In general, there is no second compensatory term. These are the terms responsible for the wake singularities for elements sketched in Fig. 3b.

The terms $\phi_2 = \log((R-V)/U)$ occur in integrations along lines U = const. They occur multiplied by different powers of U. Terms $U^n \log U$ can be considered as constants of integration which cancel (even for n = 0 and in the limit $U \to 0$). Therefore, it suffices if we discuss expression $U^n \log (R-V)$. One best returns to the original coordinates u = -U, v = -V. For a point to the line U = 0, one must evaluate $(-U)^n[\log(R_2+v_2)-\log(R_1+v_1)]$. If the point (x,y) lies close to the line u = const and $v > v_2 > v_1$, then $R_2+v_2 \approx 2R_2$; $R_1+v_1 \approx 2R_1$ the terms are not singular. For $v < v_1$ one writes $\log(R_1+v_1) = \log(\frac{u^2}{R-v_1})$, if $v < v_2$ the singular term $\log u^2$ is canceled by the contribution of point 2. For $v_1 < v < v_2$, this cancellation does not occur. The terms ϕ_2 express singularities in the upwash along element sides for which $\alpha \neq \pi/2$.

According to these considerations one must examine how the formulae for the integrals derived from ϕ_1 (combined into a vector ψ_1) appear in the limit $Y \to 0$, and how the formulae for ϕ_2 appear in the limit U = 0. One should always obtain finite expressions.

Next, consider ϕ_1 in the case $\alpha \neq \pi/2$, $\delta \neq \pi/2$. The origin lies at a point $\xi_i \eta_i$; one integrates along a line p = const from q_j

to q_{j+1}. Singularities in the vector ψ arise if one of the endpoints of the interval of integration lies at $Y = 0$. The components of the vector ψ_1 are the integrals shown in Eq. (181). The term which becomes singular for $Y = 0$ is $\log(k(R-X)$, it behaves as $\log Y)$. The factor $q^n - (ptg\gamma)^n$ of this term vanishes as $(Y\cos^{-1}\gamma)$. The term is simply disregarded if $Y = 0$ at one of the limits. For $\gamma = \pi/2$, one has $p = -Y$. The expressions ψ_1 are then found in Eqs. (184). They do not vanish. But the element of the diagonal matrix with powers of $p = -Y$ now vanish. Again one simply disregards this limit.

Similar considerations apply to ϕ_2 (and therefore ψ_2). For $\delta \neq 0$ the singular element $\log (R-V)/U$ in Eqs. (182) vanishes if $V = 0$ at one of the limits. For $\delta \neq 0$, $p = u$, and the diagonal matrix with elements given by powers of p vanishes.

Now we identify the singularities due to $\phi_3 = Y^{-1}$ and $\phi_4 = RY^{-1}$ which appear after the integration with respect to x and y has been carried out.

We begin with cases $\alpha = \pi/2$.

Then one has $\beta^2 X\phi_3$ and $\beta^2 R\phi_4$ as contributions to $I_{00}(\pi/2)$; and $(3\beta^2/2)X^2\phi_3$ and $(3\beta^2/2)XR\phi_4$ as contributions to $\bar{I}^0_{10}(\pi/2)$.

Discussion for ϕ_3.

Case $\gamma = \pi/2$.

One remembers that the areas of integration over x and y are triangles. Therefore one forms

$$-\iint X\phi_3 dxdy = -\iint XY^{-1} dxdy$$

$$= -\frac{1}{2}\int_j^{j+1} Xdx = -(1/4)X^2 \Big|_j^{j+1}$$

The negative sign arises because the convention that one integrates around the xy-element in the counterclockwise sense. If the integration extends from point j to point j+1 in Fig. 14 then the element lies above the line j, j+1 and the contribution of the triangle must be subtracted. In the notation of Eq. (178) one then has

$$J^{0}_{00,3}\ (\pi/2,\ \pi/2) = -(\beta^{2}/4)X^{2}$$

Similarly

$$J^{0}_{10,3}\ (\pi/2,\ \pi/2) = -(\beta^{2}/6)X^{3}$$

In spite of the fact that the area of the triangle tends to zero, the integral over the triangle is finite. The same behavior will be found for $J_{..,4}(\pi/2,\pi/2)$.

For ϕ_3 and $\gamma \neq \pi/2$ one has

$$J^{0}_{00,3}(\pi/2,\gamma) = \beta^{2}(p/2) \int XY^{-1}dq$$

$$= \beta^{2}(p/2)\int(q\sin\delta + p\cos\delta)(q\cos\delta - p\sin\delta)^{-1}dq$$

Hence,

$$J^{0}_{00,3}(\pi/2,\gamma) = (\beta^{2}/2)[\mathrm{tg}\gamma pq + p^{2}\cos^{-2}\gamma\log(k|Y|)]$$

If one of the limits of integration should be $Y = 0$, then the last term gives a logarithmic singularity. For $y = 0$, one has $p^{2}\cos^{-2}\gamma = X^{2} + O(y)$. Thus one can write

$$J^{0}_{00,3}(\pi/2,\gamma) = (\beta^{2}/2)[\mathrm{tg}\gamma pq + (X^{2} + O(Y)\log(k|Y|))$$

This displays the singularity in terms of X and Y. Of course, in all applications these infinities must ultimately cancel. The only purpose to retain them in the computations as separate terms is to provide a check.

For the contribution of ϕ_3 to $J^0_{10}(\pi/2,\gamma)$ one must evaluate

$$J^0_{10,3}(\pi/2,\gamma) = (3\beta^2/2)(p/3)\int X^2Y^{-1}dq$$

$$= (3\beta^2/2)(p/3)\int \phi_3(\sin^2\gamma q^2 + 2\sin\gamma\cos\gamma pq + \cos^2\gamma p^2)dq$$

The individual integrals are found in Eq. (183). Singularities arise because of the factor log (kY) if one of the limits is Y = 0. Collecting the terms contributing to the singularity one obtains

$$(3\beta^2/2)(p^3/3)\cos^{-1}\gamma[\sin^2\gamma tg^2\gamma + 2\sin\gamma\cos\gamma tg\gamma + \cos^2\gamma)\log(ky)$$

$$= (3\beta^2/2)(p^3/3)\cos\gamma(tg^4\gamma + 2tg^2\gamma+1) = (3\beta^2/2)(1/3)(p/\cos\gamma)^3\log(kY)$$

Thus, substituting the remaining terms of Eqs. (183)

$$J^0_{10,3}(\pi/2,\gamma) =$$

$$(\beta^2/2)\{(\sin\gamma tg(\gamma/2)pq^2 + p^2q\sin\gamma(2 + tg^2\gamma) + (X^3 + O(Y))\log(k/Y)\}$$

This terminates the evaluation of $I_{..,3}$ for $\alpha = \pi/2$. (The results can be summarized by stating that it suffices that one disregards the singular terms, provided of course, that one uses analytical expressions in which they cancel on theoretical grounds.)

Now we discuss the corresponding terms for ϕ_4 for $\alpha = \pi/2$

$$J^0_{00,4}(\pi/2,\pi/2) = -\beta^2/2)\int Rdx = -(\beta^2/4)[RX-Y^2\log(k(R-X)]$$

For $Y \to 0$, $X > 0$, we write $\log k(R-X) = \log kY^2-\log(R+X)$. For $Y = 0$, the expression remains finite (although the area of the triangle is zero) one obtains for $\gamma = \pi/2$

$$J^0_{00,4}(\pi/2,\pi/2) = -(\beta^2/4)RX \quad \text{for } Y = 0,$$

$$J^0_{10,4}(\pi/2),\pi/2) = -(3\beta^2/2)(1/3) \int XRdX$$

$$J^0_{10,4}(\pi/2,\pi/2) = -(\beta^2/6)R^3$$

For $\gamma \neq \pi/2$, one has to evaluate

$$J^0_{00,4} = (p/2) \int \phi_4 dq$$

This is one of the exprєssion Eq. (183). The individual terms on the right of Eq. (183) are found in Eq. (180). After substitution one obtains

$$J^0_{004}(\pi/2,\gamma) =$$

$$= (\beta^2/2)\{p^2\cos^{-2}\gamma\log((R-X)/Y) + \cos^{-1}\gamma[pR - p^2 tg\gamma\log k(R-q)]\}$$

$$= (\beta^2/2)\{((X^2+0(Y))(\log(R-X)-\log kY)+\cos^{-1}\gamma[pR-p^2 tg\gamma\log(k(R-q)]\}$$

$$= (\beta^2/2)\{(X^2+0(Y))(\log(kY)-\log(k(R+X)+\cos^{-1}\gamma[pR-p^2 tg\gamma\log(k(R-q)]\}$$

The first version is the general formula, the second version dislays the singularity for $X < 0$, and the third version the singularity of $X > 0$. The last term is singular for $p = 0$, but then it vanishes because of the factor p^2. Next we evaluate

$$J^0_{10,4}(\pi/2,\gamma)$$

Next we evaluate

$$J^0_{10,4}(\pi/2,\gamma) = (3\beta^2/2)(p/3) \int X\phi_4 dq$$

$$= (\beta^2/2)[\cos\gamma p^2 \int \phi_4 dq + \sin\gamma p \int q\phi_4 dq]$$

Here expressions from Eq. (183) are substituted:

$$J_{10,4}(\pi/2,\gamma) =$$

$$= (\beta^2/2)\{p^4\cos^{-3}\gamma \int R^{-1}Y^{-1}dy + ptg\gamma \int R^{-1}q^2dq + p^2\cos^{-2}\gamma \int R^{-1}qdq$$

$$+ p^3tg\gamma(1 + \cos^{-2}\gamma)\int R^{-1}dq\}$$

Only the first term within the braces has a denominator Y, which will give a singularity for $Y = 0$. Substituting expressions from Eq. (180) one obtains

$$J_{10,4}(\pi/2,\beta) = \beta^2/2\{p^3\cos^{-3}\gamma\log(R-X)/Y) + (ptg\gamma/2)(Rq+p^2\log(k(R-q))$$

$$+ p^2\cos^{-2}\gamma R - p^3tg\gamma(1 + \cos^{-2}\gamma)\log(kR-q))\}$$

The singularity for $Y = 0$ is displayed by writing

$$p^3\cos^{-2}\gamma = X^3 + O(Y)$$

Then

$$p^3\cos^{-3}\gamma\log(R-X/Y) = (X^3 + O(Y)(\log(k(R-X) - \log(kY))$$

$$= (X^3 + O(Y)(-\log(k(R+X) + \log(k/Y))$$

The first form is suitable if $X < 0$, the second if $X > 0$. For $\alpha \neq \pi/2$ one finds terms with ϕ_3 and ϕ_4 in I_{10}^0, namely

$$\beta^2\cos^{-2}\alpha\ U^2\phi_3 = \beta^2\cos^{-2}\alpha\ U\ Y^{-1} \text{ and } \beta^2\cos^{-1}\alpha\ U\phi_4 = \beta^2\cos^{-1}\alpha\ URY^{-1}$$

For $\gamma = \pi/2$, one has

$$\delta = \gamma-\alpha = (\pi/2 - \alpha)$$

and

$$p = -Y, \; q = X$$

Then

$$U = -Y \sin\alpha + X \cos\alpha$$

$$U^2 = Y^2 \sin^2\alpha - 2XY \sin\alpha \cos\alpha + X^2 \cos^2\alpha$$

Hence

$$J^0_{10,3}(\alpha,\pi/2) = \beta^2 \cos^{-2}\alpha(p/3) \int U^2 \phi_3 dq$$

$$= (\beta^2/3)\cos^{-2}\alpha[-\sin^2\alpha Y^2 \int dx + 2\sin\alpha\cos\alpha Y \int Xdx - \cos^2\alpha \int X^2 dx$$

$$J^0_{10,3}(\alpha,\pi/2) = (\beta^2/3)[-tg^2\alpha Y^2 X + tg\alpha Y X^2 - (\cos^2\alpha/3)X^3]$$

Here no singularity for $Y = 0$ occurs; the expression does not vanish, even though the area of the triangle from which it arises is zero.

Next we discuss

$$J^0_{10,4}(\alpha,\pi/2) = (\beta^2/3)p \cos^{-1}\alpha \; U\phi_4 dq$$

$$= (\beta^2/3)\cos^{-1}\alpha[Y\sin\alpha \int RdX - \cos\alpha \int RXdx]$$

$$J^0_{10,4}(\alpha,\pi/2) = (\beta^2/3)\{((tg\alpha)/2)Y[RX - Y\log(k(R-X)] - \frac{1}{3} R^3\}$$

The singularity which is present for $X > 0$, is of the character $Y^3 \log Y$. For $Y = 0$ one obtains

$$J^0_{10,4}(\alpha,\pi/2) = -(\beta^2/9)R^3$$

For $\gamma \neq \pi/2$ one has

$$J^0_{10}(\alpha,\gamma) = \beta^2(p/3)\cos^{-2}\alpha \int U^2\phi_3 dq$$

$$= \beta^2(p/3)\cos^{-2}\alpha \, \{\cos^2\delta \; p^2 \int \phi_3 dq + 2\sin\delta\cos\delta \; p \int \phi_3 q dq$$

$$+ \sin^2\delta \int \phi_3 q^2 dq\}$$

Here expressions from Eq. (183) are substituted. We collect terms of the same character; the terms logk(Y) is of particular interest

$$J^0_{10,3}(\alpha,\gamma) =$$

$$(\beta^2/3)\cos^{-2}\alpha\cos^{-1}\gamma\{(2\sin\delta\cos\delta + \sin^2\delta tg\delta)p^2q + ((\sin^2\delta)/2)pq^2$$

$$+ p^3(\cos^2\delta + 2\sin\delta\cos\delta tg\gamma + \sin^2\delta tg^2\gamma)\log(kY)\}$$

The factor of log kY simplifies to

$$p^3\cos^2\delta(1 + 2tg\delta tg\gamma + tg\delta^2 tg^2\gamma) = \cos^2\delta(1 + tg\delta tg\gamma)^2$$

$$= p^3\cos^{-2}\gamma\cos(\gamma-\delta)^2$$

But

$$\gamma - \delta = \alpha$$

Thus one finds

$$J^0_{10,3}(\alpha,\gamma) = (\beta^2/3)\{\cos^{-2}\alpha\cos^{-1}\gamma[\sin\delta\cos\delta(2 + tg\delta tg\delta)p^2q$$

$$+ (1/2)\sin^2\delta pq^2] + p^3\cos^{-3}\gamma\log(kY)\}$$

One has $p^3\cos^{-3}\gamma = X^3 + O(Y)$.

This shows the character of the singular term; for $Y \to 0$ it behaves as $(\beta^2/3)X^3\log(kY)$.

Consider finally

$$J^0_{10,4}(\alpha,\gamma) = \beta^2\cos^{-1}\alpha(p/3)\int U\phi_4 dq$$

$$= \beta^2\cos^{-1}\alpha(p/3)[\cos\delta\ p\int\phi_4 dq + \sin\delta\int q\phi_4 dq]$$

Here the expressions from Eq. (183) are substituted. One then obtains

$$J^0_{10,4}(\alpha,\gamma) = (\beta^2/3)\cos^{-1}\alpha\{p^4\cos^{-3}\gamma(\cos(\delta-\gamma)\int R^{-1}y^{-1}dq$$

$$+ p^2\cos^{-2}\gamma\cos(\delta-\alpha)\int qR^{-1}dq$$

$$+ p^3\cos^{-3}\gamma\cos\delta(\sin\gamma\cos\gamma + 1)\int R^{-1}dq$$

$$+ \cos^{-1}\gamma\ \sin\delta\ p\int R^{-1}q^2 dq\}$$

Here expressions from Eq. (180) and $\delta-\gamma = \alpha$ are substituted

$$J^0_{10,4}(\alpha,\gamma) = (\beta^2/3)\{p^3\cos^{-3}\gamma\log(R-X)/Y)]$$

$$+ p^2\cos^{-2}\gamma R-p^3\cos^{-1}\alpha\cos^{-3}\gamma\cos\delta(\sin\gamma\cos\gamma+1)\log(k(R-q))$$

$$+ (1/2)\cos^{-1}\alpha\cos^{-1}\gamma\sin\delta[Rpq + p^3\log(k(R-q))]\}$$

Here again $p^3\cos^{-3}\gamma = X^3 + O(Y)$, so that the singular term due to $\log(R-X/Y)$ is readily recognized.

SECTION VIII
CONCLUDING SURVEY

The sheer volume of details and formulae may make it difficult to extract from this report the information needed for numerical work. We describe here the main concepts, the definitions and list the important equations.

One observes that the formulae become longer (although not really more complex) as one takes higher terms in the development with respect to the reduced frequency into account. But this applies only if the xy-element coincides with, or lies in, the vicinity of the $\xi\eta$-element. For elements at a distance the different terms of the development with respect to frequency are lumped together, one obtains rather smooth functions, the integrations are carried out numerically and this difficulty does not arise. In any case one must choose a subdivision of the wing into elements so that the linear elemental pressure distribution give a sufficient accuracy. This implies that for an xy-element in the neighborhood of an $\xi\eta$-element the terms of low order in the development with respect to the reduced frequency will suffice. Most likely the powers zero and one will be enough. (For elements at a greater distance one should go further in the development. But we mentioned already that then the integrations can be carried out without difficulty.)

The overall arrangement of the computation is symbolized by Eq. (7). The definitions of the "housekeeping" matrices $M^{(1)}$ and $M^{(3)}$ are given in the paragraph preceding Eq. (7). An element of the matrix $M^{(2)}$ gives the average upwash in some xy-element due to one elemental pressure distribution. (In each triangular $\xi\eta$-element there are three elemental pressure distributions.)

The elemental pressure distributions are expressed in terms of coordinates ξ and η; the coordinates within an upwash element are x and y. The pressure difference between the upper and lower side of the wing is denoted by $\Delta p(\xi,\eta)$, to make analytical integrations possible one must consider $\bar{\Delta}p(\xi,\eta) = \Delta p(\xi,\eta)\exp(ik\xi)$ as unknown functions (Eq. (6)).

To each triangular element there belong three elemental shape functions, shown in Eq. (11). They are linear functions which assume the value one at the corner of the triangle with subscript j and zero at the other corners.

For an xy-element at a sufficient distance from the $\xi\eta$-element one substitutes the expressions Eq. (11) into Eq. (10) and carries out the integrations numerically. To make analytical integrations feasible Eq. (10) contains a weight factor exp(ikx).

For xy-elements at a distance from the $\xi\eta$-element but within its wake or close to it, one applies the procedure given by Eqs. (20) through (31) (if one restricts oneself to the powers zero and one of the reduced frequency and by Eqs. (20) through (42) for the general case). In the equations (without number) starting after Eq. (42) and proceeding to the end of the section the procedure is specialized to triangular elements and carried out analytically. One stops with the evaluation of f_4 if the powers of the reduced frequency are only zero and one.

Before the integrals of the upwash over an xy-element close to a $\xi\eta$-element are evaluated, we have derived formulae for the upwash at a given fixed point x,y. These results are of an intermediate nature. There is no need to evaluate these formulae unless one wants to show the details of the upwash distribution due to one elemental pressure distribution. We describe these results because they form an important part of the overall procedure. The pressure difference $\bar{\Delta}p$ are written in the form Eq. (51). The functions $c_{mn}(x,y)$ are expressed in terms of the elemental pressure distribution given by Eq. (11). (In practice only the combinations (0,0), (1,0), and (0,1) for (m,n) are used.)

If one chooses to work only with (m,n) = (0,0), then one has elements with constant $\bar{\Delta}p$, and if it is not possible to maintain continuity as one moves from one element to its neighbor. Then if the $\xi\eta$-element has a side parallel to the ξ axis, one has at the wake line originating at this side $(\eta-y)^{-1}$ singularity in the upwash. To obtain finite averages for the upwash, the upwash areas (over which the averages are formed) must overlap these wake lines. If none of the sides of the $\xi\eta$-element is parallel to the ξ axis, then the singularities are of the character $\log(\eta-y)$ and the average remains finite without an overlap. One can then develop a method in which the pressure and upwash areas coincide with the pressure elements.

The general procedure obtained after a limiting process $z \to 0$ has been carried out, is summarized in the remarks following Eq. (88a). The case $m = 0$, $n = 0$ is exceptional, the procedure is given by Eq. (88). To obtain the upwash at a fixed point xy, one has to carry out integrations around the contour of the $\xi\eta$-element, separately for each individual term in the representation for the pressure.

Carrying out the procedure one obtains definite integral over the sides of the triangle, denoted by ${}^{i,i+1}\bar{I}^{\ell}_{m,n}$. The subscripts (here m and n) refer to the term in the development of $\bar{\Delta}p$, the superscripts on the left (here i and i+1) refer to the numbering of the corners of the $\xi\eta$-element. The indefinite integral is denoted by $I^{\ell}_{m,n}$. The slope of the side enters by Eq. (107)

$$tg\alpha_{i,i+1} = (\xi_{i+1} - \xi_i)/(\eta_{i+1} - \eta_i)$$

The indefinite integral depends upon

$$X = x-\xi$$
$$Y = y-\eta$$

and

$$\alpha_{i,i+1}$$

Furthermore, we have used

$$R = (X^2 + Y^2)^{1/2}$$

$$U = X \cos\alpha - Y \sin\alpha$$

$$V = X \sin\alpha - Y \cos\alpha$$

(see Eq. (140)).

The indefinite integrals are denoted by $I_{m,n}^{(\ell)}$ (X,Y,α). Here the limits of ξ and η for the individual sides of the triangle must be substituted. One obtains a number of functions, that depend upon $x-\xi_i$, $\eta-y_i$, and $\alpha_{i,i+1}$. The expression $I^{\ell}_{m,n}$ are divided because of the different analytical characters of the expressions, for instance

$$I_{00}^{0} = I_{00}^{01} + I_{00}^{02}$$

The results are found in Eqs. (114) through (120) for $\alpha = \pi/2$, and Eqs. (141), (147) through (155).

Formulae for the integration with respect to x and y are derived in Section VI. The integrals appear now as contour integrals around the xy-element. A summary of the results before this integration is found in Eqs. (167), (168), and (169), they are a repetition in a more condensed form of the equations listed above, except for the introduction of $\tilde{I}_{10}$ and $\tilde{I}_{01}$, defined by Eq. (157).

In the integration the slope of the sides of the xy-element appears given by Eq. (161)

$$tg\alpha_{i,i+1} = \frac{x_i - x_{i+1}}{y_i - y_{i+1}}$$

The angle α defined for the sides of the $\xi\eta$-element appears again. We have furthermore (Eq. (164a))

$$\delta = \gamma - \alpha$$

Regarding this last phase of the computation we refer to the description given in Section VI.

In evaluating these expressions one will find infinities. These and other singularities are discussed in Section VII. Critical, of course, are the lines of the wake that start at the corners of the $\xi\eta$-elements, but also, although the singularities are less pronounced, the sides of the $\xi\eta$-triangle and their straight extensions through the $\xi\eta$-plane. For methods in which the effect of such infinities cancel on theoretical grounds, they can simply be disregarded, when they appear during the computations.

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APPENDIX A
THE BASIC EQUATION

The classical equation for oscillatory flows is rederived here as a convenience to the nonspecialist who would like to know how the equation comes about, but does not want to go back to the original literature.

The formulation uses the accleration potential. Usually, and also in the present case, the acceleration potential is used in connection with the linearized flow equations. But according to an observation (which was made to the author by Ernst Hoelder) it has also a meaning in a nonlinearized isentropic flow; it is given by the negative enthalpy of the particle.

The following derivation is made for the linearized flow equation. Denoting by p, ρ, and ψ, respectively, the deviation of the pressure from the free stream pressure, the free stream density, and the acceleration potential, one has

$$p = -\rho\psi \tag{A.1}$$

In a linearized flow with free stream velocity U, and velocity components U + u, v, and w in the x, y, and z directions*, respectively, of a Cartesian system of coordinates, one has the components of the acceleration

$$(\partial u/\partial t) + U(\partial u/\partial x),$$

$$(\partial v/\partial t) + U(\partial v/\partial x),$$

and

$$(\partial w/\partial t) + U(\partial w/\partial x).$$

Expressed in terms of a velocity potential, ϕ, the velocity components are

---

*for simplicity we use in this Appendix x, y, z, etc., before a Prandtl-Glauert coordinate transformation has been carried out although in the main body of the report the notation $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ is used.

$$u = (\partial\phi/\partial x),\ v = \partial\phi/\partial y,\ \text{and}\ w = \partial\phi/\partial x$$

Therefore,

$$\psi = \frac{\partial\phi}{\delta t} + U\frac{\partial\phi}{\partial x}$$

The velocity potential is then expressed in terms of an acceleration potential $\psi(x,y,z,t)$

$$\phi(x,y,z,t) = U^{-1}\int_{-\infty}^{x} \psi(\xi,y,z,t + \frac{\xi - x}{U})d\xi$$

(The velocity potential will be used in an intermediate step to evaluate the upwash due to a given acceleration potential.) One has indeed

$$U(\partial\phi/\partial x) = \psi(x,y,z,t) - U^{-1}\int_{-\infty}^{x} \psi_t(\xi,y,z,t + \frac{\xi - x}{U})d\xi$$
$$= \psi(x,y,z,t) - \partial\phi/\partial t$$

Choosing

$$\phi(x,y,z,t) = \tilde{\phi}(x,y,z)\exp(i\nu t)$$
$$\psi(x,y,z,t) = \tilde{\psi}(x,y,z)\exp(i\nu t)$$

One obtains

$$\tilde{\phi}(x,y,z) = U^{-1}\int_{-\infty}^{x} \tilde{\psi}(\xi,y,z)\exp(i\nu\frac{\xi - x}{U})d\xi$$

ϕ and ψ are made dimensionless with UL and U^2, respectively, where L is some characteristic length.

$$\tilde{\phi} = UL\bar{\phi}$$
$$\tilde{\psi} = U^2\bar{\psi}$$

Furthermore

$$\bar{x} = x/L,\ \bar{y} = y/L,\ \bar{z} = z/L,\ \text{and}\ \nu = \frac{kU}{L}$$

Then one has

$$\tilde{\phi}(\bar{x}L,\bar{y}L,\bar{z}L) = UL\bar{\phi}(\bar{x},\bar{y},\bar{z})$$
$$\tilde{\psi}(\bar{x}L,\bar{y}L,\bar{z}L) = U^2\bar{\psi}(\bar{x},\bar{y},\bar{z}) \qquad (A.2)$$

and one obtains

$$\bar{\phi}(x,y,z) = \int_{-\infty}^{\bar{x}} \bar{\psi}(\xi,\bar{y},\bar{z})\exp(ik(\xi-\bar{x}))d\xi \qquad (A.3)$$

One has, from Eq. (A.1)

$$p = -\rho U^2\bar{\psi}$$

Setting

$$\bar{p} = \Delta p/(\rho U^2/2) \qquad (A.4)$$

one obtains

$$\bar{p} = -2\bar{\psi} \qquad (A.5)$$

From now on the bars will be omitted.

ϕ and ψ satisfy the same linearized differential equation. Originally,

$$(1 - M^2)\psi_{xx} + \psi_{yy} + \psi_{zz} - \frac{2U}{a^2}\psi_{xt} - \frac{1}{a^2}\psi_{tt} = 0 \qquad (A.6)$$

(where a is the free stream velocity of sound and $M = U/a$ is the Mach number). Introducing nondimensional quantities one obtains for the oscillatory case (after omission of the bars)

$$(1 - M^2)\psi_{xx} + \psi_{yy} + \psi_{zz} - 2ik\,M^2\,\psi_x + k^2M^2\psi = 0 \qquad (A.7)$$

Let

$$1 - M^2 = \beta^2$$
$$r^2 = y^2 + z^2 \qquad (A.8)$$
$$R^2 = x^2 + \beta^2 r^2$$

A solution of Eq. (A.7) for outgoing waves which has an R^{-1} singularity at the origin is given by

$$\psi_{source} = - R^{-1} \exp\left[\frac{ikM}{\beta^2} (Mx - R)\right] \qquad (A.9)$$

Eq. (A.9) can be verified by substitution into Eq. (A.7). If the expression would be used for the potential it would represent an oscillating source. The expression for ψ to be used here is given by

$$\psi = -\partial\psi_{source}/\partial z$$

One has

$$\psi = - z\left(\frac{\beta^2}{R^3} + \frac{Mik}{R}\right) \exp\left[\frac{ikM}{\beta^2}(Mx - R)\right] \qquad (A.10)$$

The potential pertaining to it is obtained from Eq. (A.3)

$$\phi(x,y,z) = -z \int_{-\infty}^{x} \left[\frac{\beta^2}{(\xi^2 + \beta^2 r^2)^{3/2}} + \frac{Mik}{(\xi^2 + \beta^2 r^2)^{1/2}}\right] \exp\left[\frac{ikM}{\beta^2} (M\xi - (\xi^2 + \beta^2 r^2)^{1/2}) + ik(\xi-x)\right]d\xi$$

or using the first of Eqs. (A.8)

$$\phi(x,y,z) = -z \exp(-ikx) \int_{-\infty}^{x} \left[\frac{\beta^2}{(\xi^2 + \beta^2 r^2)^{3/2}} + \frac{Mik}{(\xi^2 + \beta^2 r^2)^{1/2}}\right] \exp\left[\frac{ik}{\beta^2}[\xi - M(\xi^2 + \beta^2 r^2)^{1/2}]\right]d\xi \qquad (A.11)$$

To simplify the argument of the exponential function, one introduces

$$v = \beta^{-2}[\xi - M(\xi^2 + \beta^2 r^2)^{1/2}] \tag{A.12}$$

and (for the upper limit of the integral)

$$V(x,r) = \beta^{-2}[x - M(x^2 + \beta^2 r^2)^{1/2}] \tag{A.13}$$

(In Ref. 1, X is used instead of V. The author has changed the notation because the variable V does not solely depend upon x.)

Then, from Eq. (A.12)

$$\xi = v + M(r^2 + v^2)^{1/2}$$

and from Eq. (A.13)

$$x = V + M(r^2 + V^2)^{1/2} \tag{A.14}$$

Hence

$$d\xi = dv\frac{(v^2 + r^2)^{1/2} + Mv}{(r^2 + v^2)^{1/2}}$$

$$R(\xi,r) = (\xi^2 + \beta^2 r^2)^{1/2} = (v^2 + r^2)^{1/2} + Mv \tag{A.15}$$

analogously

$$R(x,r) = (x^2 + \beta^2 r^2)^{1/2} = (V^2 + r^2)^{1/2} + MV \tag{A.15a}$$

Then by substitution into Eq. (A.11)

$$\phi(x,y,z) =$$

$$-z\,\exp(-ikx)\int_{-\infty}^{V} \exp(ikv)\left[\frac{\beta^2}{[(v^2 + r^2)^{1/2} + Mv]^2(r^2 + v^2)^{1/2}}\right.$$

$$\left. + \frac{ikM}{[(v^2 + r^2)^{1/2} + Mv](r^2 + v^2)^{1/2}}\right] dv$$

In the second term of the bracket in the integrand an integration by parts is carried out:

$$\phi(x,y,z) = -z\exp(-ikx)\left\{\left.\frac{M\exp(ikv)}{[(v^2+r^2)^{1/2}+Mv](v^2+r^2)^{1/2}}\right|_{-\infty}^{V}\right.$$
$$+\int_{-\infty}^{V}\exp(ikv)dv\left[\frac{1-M^2}{[(v^2+r^2)^{1/2}+Mv]^2(v^2+r^2)^{1/2}}\right.$$
$$\left.\left.-\frac{d}{dv}\frac{M}{[(v^2+r^2)^{1/2}+Mv](v^2+r^2)^{1/2}}\right]\right\}$$

The contribution of the lower limit in the term outside of the integral vanishes. With Eq. (A.15) one obtains for the upper limit

$$\frac{M\exp(ikV)}{R(x,r)(V^2+r^2)^{1/2}}$$

The integral simplifies to

$$\int_{-\infty}^{V}\frac{\exp(ikv)}{(v^2+r^2)^{3/2}}\,dv$$

Thus, one finally finds for the velocity potential that pertains to the acceleration potential given by Eq. (A.10)

$$\phi(x,y,z) = -z\exp(-ikx)\left[\frac{M\exp(ikV)}{R(x,r)(V^2+r^2)^{1/2}}+\int_{-\infty}^{V}\frac{\exp(ikv)}{(v^2+r^2)^{3/2}}\,dv\right] \tag{A.16}$$

Where V is defined in Eq. (A.13), r and R in Eq. (A.8). The upwash is then given by $\partial\phi/\partial z$. Eq. (A.10) is a fundamental sclution with the singular point at the origin. The flow field is represented by a superposition of such fundamental solutions but with singular points (ξ,η) lying in the plane of the wing. One then must replace x, by $x-\xi$, and y by $y-\eta$. Consequently, one has now

$$
\begin{aligned}
r^2 &= [(y-\eta)^2 + z^2] \\
R &= [(x-\xi)^2 + \beta^2 r^2]^{1/2} \\
V &= \beta^{-2}[(x-\xi) - MR]
\end{aligned}
\tag{A.17}
$$

We note that

$$
(V^2 + r^2)^{1/2} = \beta^{-2}(R - M(x-\xi)) \tag{A.18}
$$

Let $f(\xi,\eta)$ be the strength of the doublets assigned to the particular solution Eq. (A.10). Then

$$
\psi(x,y,z) = -z \int_A d\xi d\eta f(\xi,\eta)\left[\frac{\beta^2}{R^3} + \frac{Mik}{R}\right] \exp\left[\frac{ikM}{\beta^2}(M(x-\xi) - R)\right] \tag{A.19}
$$

Here A is the wing area. We determine $\lim_{z\to 0} (\psi(x,y,z)$. The limit is obviously zero, as long as $R \neq 0$, for instance for all points outside of the wing. Assuming that the origin lies within the wing area we evaluate (for simplicity) the expression for $x = 0$, $y = 0$, $z = \varepsilon > 0$), i.e., we approach the plane of the wing from above. Accordingly we consider

$$
\begin{aligned}
\psi(0,0,\varepsilon) = -\varepsilon \int_A d\xi d\eta f(\xi,\eta) & \left[\frac{\beta^2}{[\xi^2 + \beta^2(\eta^2 + \varepsilon^2)]^{3/2}}\right. \\
& \left. + \frac{Mik}{[\xi^2 + \beta^2(\eta^2 + \varepsilon^2)]^{1/2}}\right] \exp\left[\frac{ikM}{\beta^2} (-M\xi - (\xi^2 + \beta^2(\eta^2 + \varepsilon^2))^{1/2}\right]
\end{aligned}
\tag{A.20}
$$

We cut out from the wing area a small ellipse, given by

$$
\xi = \gamma \cos \alpha, \ \eta = \gamma\, \beta^{-1} \sin \alpha
$$

where γ is constant and independent of ε. For points outside of this ellipse, the integrand is bounded; because of the factor ε in front the expression vanishes for $\varepsilon = 0$. Retaining within the ellipse only the dominant terms, one finds

$$\lim_{\varepsilon \to 0} \psi(0,0,\varepsilon) = -\lim_{\varepsilon \to 0} \beta^2 f(0,0) \int_{\text{ellipse}} \frac{\varepsilon \, d\xi d\eta}{(\xi^2 + \beta^2(\eta^2 + \varepsilon^2))^{3/2}}$$

In evaluating the integral we set

$$\xi = \varepsilon\tilde{\xi}, \; \eta = \varepsilon\tilde{\eta}$$

The boundary of the region then becomes

$$\tilde{\xi} = \frac{\gamma}{\varepsilon} \cos \alpha, \; \tilde{\eta} = \frac{\gamma}{\varepsilon} \beta^{-1} \sin \alpha$$

and one obtains

$$\lim_{\varepsilon \to 0} \psi(0,0,\varepsilon) = -\lim_{\varepsilon \to 0} \beta^2 f(0,0) \int_{\text{ellipse}} \frac{d\tilde{\xi} d\tilde{\eta}}{(\xi^2 + \beta^2(\tilde{\eta}^2 + 1))^{3/2}}$$

Now we set

$$\tilde{\xi} = \tilde{r} \cos \alpha$$

$$\tilde{\eta} = \frac{\tilde{r}}{\beta} \sin \alpha$$

The boundary of the region is then given by

$$\tilde{r} = \gamma/\varepsilon$$

Moreover

$$d\tilde{\xi} d\tilde{\eta} = \frac{\tilde{r}}{\beta} \, d\tilde{r} d\alpha$$

Then

$$\lim_{\varepsilon \to 0} \psi(0,0,\varepsilon) = -\lim_{\varepsilon \to 0} 2\pi\beta \; f(0,0) \int_0^{\gamma/\varepsilon} \frac{\tilde{r} d\tilde{r}}{(r^2 + \beta^2)^{3/2}}$$

$$= \lim_{\varepsilon \to 0} 2\pi\beta \; f(0,0)(r^2 + \beta^2)^{-1/2} \Big|_0^{\gamma/\varepsilon} = - 2\pi \; f(0,0)$$

The acceleration potential Eq. (A.20), therefore, gives at point (ξ,η) the value $-2\pi\ f(\xi,\eta)$ and for the nondimensional pressure defined in Eq. (A.5) $4\pi f(\xi,\eta)$. The contribution to the lift is the pressure difference between the lower and the upper side. It is denoted by Δp. Thus one finds

$$\Delta\bar{p} = -\ 8\pi f(\xi,\eta)$$

The upwash due to this lift is then found from a corresponding superposition of expressions Eq. (A.16)

$$w(x,y,z) = \frac{1}{8\pi}\frac{\partial}{\partial z}\left[z\int_A \Delta p(\xi,\eta)K(x-\xi,\ y-\eta,\ z,k)d\xi d\eta\right] \tag{A.21}$$

where

$$K = \exp(-ik(x-\xi)\left[\frac{M\exp(ikV)}{R(x-\xi,r)(V(x-\xi,r)^2 + r^2)^{1/2}} + \int_{-\infty}^{V}\frac{\exp(ikv)}{(v^2 + r^2)^{3/2}}\,dv\right] \tag{A.22}$$

r, R and V, and $(v^2 + r^2)$ are given in Eqs. (A.17) and (A.18). This equation is due to Kuessner it holds throughout the flow field. Identifying $w(x,y,0)$ with the upwash found from the boundary conditions at the wing surface, $z = 0$, one obtains an integral equation for $\Delta p(\xi,\eta)$. Frequently this is written as

$$w(x,y,0) = \frac{1}{8\pi}\int_A \Delta p(\xi,\eta)\ K(\xi-x)(\eta-y,k)d\xi d\eta$$

i.e., one makes the limit $z\to 0$ immediately. Because of the singularities of K, this expression must then be interpreted by some special technique (e.g., the one due to Mangler), but then one has to verify that the assumptions made by Mangler are applicable. The present analysis is based on Eqs. (A.21) and

(A.22). The evaluation of the kernel K as a development with respect to k is due to Ueda (Ref. 2). (See Appendix B.)

APPENDIX B
REDERIVATION OF SOME FORMULAE DUE TO UEDA

The derivation of the formulae on which the present report is based seem to be available only in Japanese (Ref. 2). At least to some readers, a rederivation may, therefore, be desirable. The results shown here are somewhat more detailed than those of Ueda. The following expression is to be evaluated:

$$B(k,\tilde{r},\tilde{V}) = \int_{-\infty}^{\tilde{V}} \frac{\exp(ikv)}{(v^2 + \tilde{r}^2)^{3/2}}\, dv \qquad (B.1)$$

A slight simplification is obtained by setting

$$\bar{k} = k\tilde{r}$$
$$\bar{V} = \tilde{V}/\tilde{r} \qquad (B.2)$$
$$v = \tilde{r}u$$

Then one obtains

$$B(k,\tilde{r},\tilde{V}) = \tilde{r}^{-2}\, \bar{B}(\bar{k},\bar{V}) \qquad (B.3)$$

with

$$\bar{B}(\bar{k},\bar{V}) = \int_{-\infty}^{\bar{V}} \frac{\exp(i\bar{k}u)}{(u^2 + 1)^{3/2}}\, du \qquad (B.4)$$

In essence the formulae are obtained by a development of the exponential function in the integrand. But, as the development generates higher and higher powers of u, the integrals will not converge as the lower limit tends to negative infinity. Therefore, we write

$$\bar{B}(\bar{k},\bar{V}) = \bar{B}_1(\bar{k}) + \bar{B}_2(\bar{k},\bar{V}) \qquad (B.5)$$

with

$$\bar{B}_1(\bar{k}) = \int_{-\infty}^{0} \frac{\exp(i\bar{k}u)}{(u^2 + 1)^{3/2}} \, du \qquad (B.6)$$

$$\bar{B}_2(\bar{k},\bar{V}) = \int_{0}^{V} \frac{\exp(i\bar{k}u)}{(u^2 + 1)^{3/2}} \, du \qquad (B.7)$$

A development of the exponential function in $\bar{B}_2$ leads to converging integrals. Real and imaginary parts must be treated separately.

$$\bar{B}_{2R}(\bar{k},\bar{V}) = \sum_{n=0}^{\infty} (-)^n \, \bar{k}^{2n} \, U_{2n}(\bar{V}) \qquad (B.8)$$

$$\bar{B}_{2I}(\bar{k},\bar{V}) = \sum_{n=0}^{\infty} (-)^n \, \bar{k}^{2n+1} \, U_{2n+1}(\bar{V}) \qquad (B.9)$$

with

$$U_n(\bar{V}) = \frac{1}{n!} \int_{0}^{\bar{V}} \frac{u^n}{(u^2 + 1)^{3/2}} \, du \qquad (B.10)$$

These integral can be expressed by elementary functions. Let W_n be the pertinent indefinite integrals including the factor $(n!)^{-1}$. Then

$$W_0(u) = u(u^2 + 1)^{-1/2} \qquad (B.11)$$

$$W_1(u) = -(u^2 + 1)^{-1/2} \qquad (B.12)$$

$$W_2(u) = (1/2)\left[\log((u^2 + 1)^{1/2} + u) - u(u^2 + 1)^{-1/2}\right] \qquad (B.13)$$

Further expressions can be obtained recursively. The relation

$$\frac{d}{du}\frac{u^{m-1}}{(u^2+1)^{1/2}} = (m-2)\frac{u^m}{(u^2+1)^{3/2}} + (m-1)\frac{u^{m-2}}{(u^2+1)^{3/2}}$$

$$= (m-2)m!(dW_m/du) + (m-1)(m-2)!(dW_{m-2}/du)$$

yields the recurrence relation

$$w_m(u) = \frac{1}{(m-2)m!}\frac{u^{m-1}}{(u^2+1)^{1/2}} - \frac{1}{m(m-2)} w_{m-2}(u) \qquad (B.14)$$

Ueda's formulae are based on this relation.

A more direct formulation, which will be used in this report, is based on formulae in Appendix C. there the following definitions (for indefinite integrals) are introduced

$$I_m^{-1/2}(w) = \int w^m(w^2+1)^{-1/2}\,dw$$
$$I_m^{-3/2}(w) = \int w^m(w^2+1)^{-3/2}\,dw \qquad (B.15)$$

They are related by

$$I_m^{-3/2}(w) = -w^{m-1}(w^2+1)^{-1/2} + (m-1)I_{m-2}^{-1/2}(w) \qquad (B.16)$$

First we evaluate B_{2I}. Setting

$$m = 2n + 1$$

one obtains

$$I_{2n+1}^{-3/2}(w) = -w^{2n}(w^2+1)^{-1/2} + 2n\,I_{2n-1}^{-1/2}(w) \qquad (B.17)$$

Furthermore, from the equation top of page 149

$$I_{2n-1}^{-1/2}(w) = \left[(2n-1)\beta_{2n-2}\right]^{-1}(w^2 + 1)^{1/2} \sum_{\ell=0}^{n-1} \beta_{2\ell}\, w^{2\ell} \qquad \text{(B.18)}$$

For the $\beta_{2\ell}$'s, one has the recurrence relation Eq. (C.8). To obtain U_{2n+1} one must substitute the limits $w = \bar{V}$ and $w = 0$ into Eqs. (B.17) and (B.18), and multiply by $((2n+1)!)^{-1}$. One obtains

$$U_1 = -(V^2 + 1)^{-1/2} - (-1) \qquad \text{(B.19)}$$

$$\begin{aligned} U_{2n+1} &= -[(2n+1)!]^{-1}\, \bar{V}^{2n}(\bar{V}^2 + 1)^{-1/2} \\ &\quad + c_{2n+1}(\bar{V}^2 + 1)^{1/2}\left(\sum_{\ell=0}^{n-1} \beta_{2\ell}\bar{V}^{2\ell}\right) - c_{2n+1};\ n \geq 1 \end{aligned} \qquad \text{(B.20)}$$

with

$$c_{2n+1} = \frac{2n}{(2n+1)!(2n-1)\beta_{2n-2}} \qquad \text{(B.21)}$$

One has the following recursion relation

$$\frac{c_{2n+1}}{c_{2n-1}} = \frac{2n}{2n-2}\,\frac{2n-3}{2n-1}\,\frac{(2n-1)!}{(2n+1)!}\,\frac{\beta_{2n-4}}{\beta_{2n-2}}$$

One has from Eq. (C.8) (setting $k = n-2$)

$$\frac{\beta_{2n-4}}{\beta_{2n-2}} = -\frac{2n-2}{2n-3}$$

Therefore

$$c_{2n+1}/c_{2n-1} = -1/(4n^2-1) \qquad \text{(B.22)}$$

Because of the factor 2n in the numerator of Eq. (B.21) the recursion starts with $n = 1$; then

$$c_3 = \frac{2}{3!} = \frac{1}{3}$$

This suffices to evaluate Eq. (B.20). But on the basis of Eq. (B.22), one can also start with $n = 0$ and $c_1 = -1$. Then Eq. (B.20) encompasses also Eq. (B.19), if the sum $\sum_{k=0}^{k=0-1}$ which then arises is understood to be zero.

The expression U_{2n+1} consists of three terms: the sum in Eq. (B.9) formed with the first of these terms gives

$$-\sum_{n=0}^{\infty} (-)^n \frac{1}{(2n+1)!} \bar{k}^{2n+1} \bar{V}^{2n}(\bar{V}^2 + 1)^{-1/2} = - \bar{V}^{-1}(\bar{V}^2 + 1)^{-1/2}\sin(\bar{k}\bar{V})$$

One thus obtains

$$\begin{aligned} \bar{B}_{2I} = &- \bar{V}^{-1}(\bar{V}^2 + 1)^{-1/2} \sin(\bar{k}\bar{V}) \\ &+ (\bar{V}^2 + 1)^{1/2} \sum_{n=1}^{\infty} (-)^n c_{2n+1} \bar{k}^{2n+1}\left(\sum_{\ell=0}^{n-1} \beta_{2\ell}\bar{V}^{2\ell}\right) \\ &- \sum_{n=0}^{\infty} (-)^n c_{2n+1} \bar{k}^{2n+1} \end{aligned} \qquad \text{(B.22a)}$$

The c_{2n+1}'s and $\beta_{2\ell}$'s are evaluated, respectively, from Eq. (B.22) with $c_1 = -1$ and from Eq. (C.8) with $\beta_0 = 1$.

To evaluate $\bar{B}_{2R}$ one needs expressions U_{2j}. They, too, are based on the formulae of Appendix C. Eq. (C.13) for $m = 2m_1$ assumes the form

$$I_{2m_1}^{-3/2} = - w^{2m_1-1}(w^2 + 1)^{-1/2} + (2m_1-1)I_{2m_1-2}^{-1/2} \qquad m_1 \geq 1$$

Examining this equation and Eqs. (C.9), (C.10), and (C.11) one finds that there is no contribution of the lower limit in Eq. (B.10). One then finds

$$U_0(\bar{V}) = -\bar{V}^{-1}(\bar{V}^2+1)^{-1/2} + \bar{V}^{-1}(\bar{V}^2+1)^{1/2} \tag{B.23}$$

$$U_2(\bar{V}) = -\bar{V}^{-1}(\bar{V}^2+1)^{-1/2}\frac{1}{2!}\bar{V}^2 + \frac{1}{2!}\log\left[(\bar{V}^2+1)^{1/2}+\bar{V}\right] \tag{B.24}$$

$$U_{2n}(\bar{V}) =$$

$$-\bar{V}^{-1}(\bar{V}^2+1)^{-1/2}\frac{V^{2n}}{(2n)!} + c_{2n}(\bar{V}^2+1)^{1/2}\sum_{\ell=1}^{n-1}\beta_{2\ell-1}\,\bar{V}^{2\ell-1}$$

$$-c_{2n}\log\left[(\bar{V}^2+1)^{1/2}+\bar{V}\right] \quad n \geq 2 \tag{B.25}$$

with

$$c_{2n} = \frac{2n-1}{(2n)!}\,\frac{1}{(2n-2)\beta_{2n-3}} \tag{B.26}$$

The c_{2j}'s are expressed explicitly in Eq. (B.27). This formula is obtained as follows:

$$c_4 = 1/16 \tag{B.26a}$$

From Eq. (C.5) which, by definition, is also satisfied by the β's one finds

$$\frac{\beta_{2n-3}}{\beta_{2n-1}} = -\frac{2n-1}{2n-2}$$

Then, one obtains from Eq. (B.26) the recurrence relation

$$c_{2n+2}/c_{2n} = -\left[2n(2n+2)\right]^{-1}$$

with the initial condition (B.26a) this is solved by

$$c_{2n} = 2(-1/4)^n\left[n!(n-1)!\right]^{-1} \tag{B.27}$$

For n=1, this gives

$$c_2 = -\frac{1}{2}$$

The last results show that Eq. (B.25) covers also Eq. (B.24) if the sum $\sum_{\ell=+1}^{\ell=0}$ is understood to be zero.

Thus one obtains

$$\begin{aligned}\bar{B}_{2R} = &- \bar{V}^{-1}(\bar{V}^2 + 1)^{-1/2} \cos(\bar{k}\bar{V}) + \bar{V}^{-1}(\bar{V}^2 + 1)^{1/2} \\ &+ 2(\bar{V}^2 + 1)^{1/2} \sum_{n=2}^{\infty} \frac{1}{n!(n-1)!} (\bar{k}/2)^{2n} \sum_{\ell=1}^{n-1} \beta_{2\ell-1} \bar{V}^{2\ell-1} \\ &- 2 \log((\bar{V}^2 + 1)^{1/2} + \bar{V}) \sum_{n=0}^{\infty} \frac{1}{(n+1)!n!} (\bar{k}/2)^{2n+2} \qquad (B.28)\end{aligned}$$

with

$$\beta_{2\ell+1}/\beta_{2\ell-1} = -\ell/(\ell + (1/2)) \quad , \quad \beta_1 = 1$$

The independent variable in $\bar{B}_1$ is $\bar{k}$. To suggest this we write temporarily

$$\begin{aligned}\bar{k} &= y \\ \bar{B}_1 &= Q\end{aligned} \qquad (B.29)$$

Then

$$Q(y) = \int_{-\infty}^{0} \frac{\exp(iyu)}{(u^2 + 1)^{3/2}} du \qquad (B.30)$$

Q(y) satisfies a differential equation closely related to the Bessel equation. One has

$$\frac{dQ}{dy} = i \int_{-\infty}^{0} \frac{u \exp(iyu)}{(1 + u^2)^{3/2}} du \tag{B.31}$$

$$\frac{d^2Q}{dy^2} = - \int_{-\infty}^{0} \frac{u^2 \exp(iyu)}{(1 + u^2)^{3/2}} du \tag{B.32}$$

d^2Q/dy^2 is not absolutely convergent; but the integral is well defined if one reformulates Eq. (B.30) as

$$Q(y) = \lim_{a \to \infty} \int_{-a}^{0} \frac{\exp(iyu)}{(1 + u^2)^{3/2}} du$$

One obtains from Eqs. (B.30) and (B.32)

$$Q - \frac{d^2Q}{dy^2} = \int_{-\infty}^{0} \frac{\exp(iyu)}{(u^2 + 1)^{1/2}} du \tag{B.33}$$

and from Eq. (B.31) by an integration by parts

$$\frac{dQ}{dy} = - i \left. \frac{\exp(iyu)}{(u^2 + 1)^{1/2}} \right|_{-\infty}^{0} - y \int_{-\infty}^{0} \frac{\exp(iyu)}{(u^2 + 1)^{1/2}} du \tag{B.34}$$

Combining Eqs. (B.33) and (B.34) one obtains the following differential equation for Q

$$\frac{d^2Q}{dy^2} - \frac{1}{y} \frac{dQ}{dy} - Q = \frac{i}{y} \tag{B.35}$$

We separate Q into ints real and imaginary parts $Q = Q_R + i\, Q_I$. Then

$$\frac{d^2Q_R}{dy^2} - \frac{1}{y} \frac{dQ_R}{dy} - Q_R = 0 \tag{B.36}$$

$$\frac{d^2Q_I}{dy^2} - \frac{1}{y} \frac{dQ_I}{dy} - Q_I = \frac{1}{y} \tag{B.37}$$

Obviously $Q(\infty) = 0$, furthermore

$$Q(0) = \int_{-\infty}^{0} \frac{du}{(u^2 + 1)^{3/2}} = \frac{u}{(u^2 + 1)^{1/2}} \Bigg|_{-\infty}^{0} = 1 \qquad \text{(B.38)}$$

These are the boundary conditions for Eq. (B.35). The solution Q_I will arise in the form of power series in y, which makes it difficult to recognize whether the boundary condition $Q(\infty) = 0$ is satisfied. It will be shown below, that the definition

$$Q_I(y) = \int_{-\infty}^{0} \frac{\sin(yu)}{(u^2 + 1)^{3/2}} du \qquad \text{(B.39)}$$

implies

$$\frac{d^2 Q_I}{dy^2} \Bigg|_{y=0} = \pi/2 \qquad \text{(B.40)}$$

The derivation is shown below. This condition will be used instead of $Q_I(\infty) = 0$. This procedure cannot be applied to Q_R; (there one would find $d^2Q_R|dy^2|_{y=0} = \infty$. From here on, the real and imaginary parts are treated separately. The following derivation of Eq. (B.40) ends with Eq. (B.47). One has

$$\frac{d^2 Q_I}{dy^2} = - \int_{-\infty}^{0} \frac{u^2 \sin(yu)}{(y^2 + 1)} du = -Q_{I1}(y) + Q_{I2}(y) \qquad \text{(B.41)}$$

with

$$Q_{I1} = \int_{-\infty}^{0} \frac{\sin(yu)}{(u^2 + 1)^{1/2}} du \qquad \text{(B.42)}$$

$$Q_{I2} = \int_{-\infty}^{0} \frac{\sin(yu)}{(u^2 + 1)^{3/2}} du \qquad \text{(B.43)}$$

It is not permissible to form the limit y = 0 under the integral sign, for within the region of integration there are always values of u for which sin(yu) is not small. We write

$$Q_{I2} = \int_{-\infty}^{-a/y} + \int_{-a/y}^{0} \frac{\sin(yu)}{(u^2 + 1)^{3/2}}\, du \quad , \quad a > 0$$

Now

$$\left| \int_{-\infty}^{-a/y} \frac{\sin(yu)}{(u^2 + 1)^{3/2}}\, du \right| < \int_{-\infty}^{-a/y} \frac{1}{(u^2 + 1)^{3/2}}\, du = \frac{u}{(u^2 + 1)^{1/2}} \Bigg|_{-\infty}^{-a/y}$$

$$= \frac{-a}{(a^2 + y^2)^{1/2}} + 1$$

This expression vanishes for $y \to 0$. Furthermore,

$$\left| \int_{-a/y}^{0} \frac{\sin(yu)}{(u^2 + 1)^{3/2}}\, du \right| < y \int_{-a/y}^{0} \frac{-u}{(u^2 + 1)^{3/2}}\, du = y(u^2 + 1)^{-1/2} \Bigg|_{u=-a/y}^{0}$$

$$= y[1 - ((a/y)^2 + 1)^{-1/2}]$$

Therefore

$$\lim_{y \to 0} Q_{I,2}(y) = 0 \qquad (B.44)$$

Q_{I1} is transformed as follows

$$Q_{I1} = \int_{-\infty}^{0} \frac{\sin(yu)}{(u^2 + 1)^{1/2}}\, du = \int_{-\infty}^{0} ((u^2 + 1)^{-1/2} + u^{-1})\sin(yu)du$$

$$- \int_{-\infty}^{0} u^{-1} \sin(yu)du \qquad (B.45)$$

The second integral can be evaluated by the calculus of residue,

$$\int_{-\infty}^{0} u^{-1} \sin(yu)du = (1/2) \int_{-\infty}^{+\infty} v^{-1}(\sin v)dv = \pi/2 \qquad (B.46)$$

The integrand in the first integral on the right of Eq. (B.45) is rewritten

$$(u^2 + 1)^{-1/2} + u^{-1} = \frac{(u^2 + 1)^{1/2} + u}{(u^2 + 1)^{1/2} u} = \frac{1}{(u^2 + 1)^{1/2} u[(u^2 + 1)^{1/2} - u]}$$

In the integrand of the following expression, one has

$$u \leq - a/y < 0$$

Therefore

$$\left| \int_{-\infty}^{-a/y} \frac{\sin(yu)du}{(u^2 + 1)^{1/2} u[(u^2 + 1)^{1/2} - u]} \right| < (y/a) \int_{-\infty}^{-a/y} \frac{du}{u^2 + 1}$$

The limit $y = 0$ of this expression is zero. Furthermore,

$$\left| \int_{-a/y}^{0} \frac{\sin(yu)du}{(u^2 + 1)^{1/2} u[(u^2 + 1)^{1/2} - u]} \right| < y \int_{-a/y}^{0} \frac{du}{u^2 + 1} \qquad (B.47)$$

Also this integral vanishes in the limit $y \to 0$. Substituting Eqs. (B.44), (B.45), and (B.46) into Eq. (B.41) one obtains, indeed, Eq. (B.40).

One notices that the homogeneous part of Eq. (B.37) is the same as in Eq. (B.36). The indicial equation for the singular point $y = 0$ in Eq. (B.37) gives the exponents 0 and 2. Therefore, there exists a homogeneous solution in the form of a power series which starts with y^2. We set for this particular solution

$$Q_3(y) = \sum_0^\infty a_{2j+2} \; y^{2j+2} \tag{B.48}$$

This gives the recurrence relation

$$(2j+2)2j \; a_{2j+2} - a_{2j} = 0$$

which is satisfied by

$$a_{2j+2} = 4(1/2)^{2j+2}[j!(j+1)!]^{-1} \tag{B.49}$$

if one chooses $a_2 = 1$. One solution of the inhomogeneous equation (B.37) can be obtained as a power series in y.

$$Q_4(y) = \sum_{j=0}^\infty (-)^j b_{2j+1} \; y^{2j+1} \tag{B.50}$$

Hence, by substitution into Eq. (B.37)

$$\sum_{j=0}^\infty (-)^j[(2j+1)(2j-1)b_{2j+1} + b_{2j-1}]y^{2j-1} = y^{-1}$$

The equation gives for j = 1

$$-b_1 y^{-1} = y^{-1}$$

Hence

$$b_1 = - 1 \tag{B.51}$$

In addition one has the recurrence relation

$$b_{2j+1}/b_{2j-1} = - (1/4)[(j+1/2)(j-1/2)]^{-1} = -[(2j)^2 - 1]^{-1} \tag{B.52}$$

Then because of Eq. (B.40)

$$Q_I(y) = Q_4(y) + \pi/4 \; Q_3(y) \tag{B.53}$$

Returning to the original notation one has

$$\bar{B}_{1I}(\bar{k}) = \sum_{j=0}^{\infty} (-)^j b_{2j+1} \bar{k}^{2j+1} + \pi \sum_{j=0}^{\infty} (\bar{k}/2)^{2j+2}[j!(j+1)!]^{-1} \tag{B.54}$$

Now

$$\bar{B}_I(\bar{V},\bar{k}) = \bar{B}_{1I}(\bar{k}) + \bar{B}_{2I}(\bar{V},\bar{k})$$

$\bar{B}_{2I}$ is found in Eq. (B.22a). The last term in Eq. (B.22a) cancels the first term in Eq. (B.54), for the b_{2j+1} and the c_{2j+1} satisfy the same recurrence relations (Eqs. (B.52) and (B.22)), the same starting values $b_1 = c_1 = -1$, and the sums have opposite signs. One thus obtains

$$\begin{aligned} \bar{B}_I = &- \bar{V}^{-1}(\bar{V}^2 + 1)^{-1/2} \sin(\bar{k}\bar{V}) \\ &+ (\bar{V}^2 + 1)^{1/2} \sum_{n=1}^{\infty} (-)^n c_{2n+1} \bar{k}^{(2n+1)}\left(\sum_{\ell=0}^{n-1} \beta_{2\ell}\bar{V}^{2\ell}\right) \\ &+ \pi \sum_{j=0}^{\infty} (\bar{k}/2)^{2j+2}[n!(n+1)!]^{-1} \end{aligned} \tag{B.55}$$

where the coefficients c_{2n+1} and the β_{2k} are obtained respectively from Eq. (B.22), with $c_1 = -1$ (or $c_3 = 1/3$), and Eq. (C.7) with $\beta_0 = 1$.

To evaluate $Q_R(y)$ we use a result about Bessel functions. Setting

$$Q_R(y) = y\ Z(y) \tag{B.56}$$

one obtains from Eq. (B.36)

$$\frac{d^2Z}{dy^2} + \frac{1}{y}\frac{dZ}{dy} - (1 + \frac{1}{y^2})Z = 0$$

which is solved by

$$Z = Z_1(iy)$$

where Z_1 is some linear combination of Bessel functions of order 1. the conditino $Z \to 0$, as $y \to 0$ is satisfied by the Hankel function of order 1, $H_1^{(1)}(iz)$. One has (see for instance Ref. 3)

$$H_\nu(z) = J_\nu(z) + i\, Y_\nu(z)$$

where

$$z = x + iy$$

$$J_1(z) = (z/2) \sum_{n=0}^{\infty} \frac{(-(z/2)^2)^n}{n!(n+1)!}$$

$$Y_1(z) = \pi^{-1} \Big\{ -(z/2)^{-1} + 2(\log(z/2))J_1(z)$$

$$- (z/2) \sum_{n=0}^{\infty} (\psi(n+1) + \psi(n+2)) \frac{(-z/2)^2)^n}{n!(n+1)!} \Big\}$$

Here

$$\psi_1 = -\gamma$$

$$\psi_n = -\gamma + \sum_{m=1}^{n-1} m^{-1}, \; n \geq 2 \qquad \text{(B.57)}$$

$$\gamma = .5772156649...$$

Then

$$\psi_{n+1} + \psi_{n+2} = -2\gamma + 2 \sum_{m=1}^{n} m^{-1} + (n+1)^{-1}$$

Observing that

$$\log(iy/2) = \log(y/2) + i\pi/2$$

one obtains

$$H_1(iy) = J_1(iy) + i\,Y_1(iy)$$

$$= \pi^{-1}\left\{-(y/2)^{-1} + \sum_{n=0}^{\infty} (\psi(n+1) + \psi(n+2) - 2\log(y/2))\,\frac{(y/2)^{2n+1}}{n!(n+1)!}\right\}$$

According to Eq. (B.56)

$$Q_R = \text{const } y\, H_1(iy)$$

and this expression satisfies the condition for $y \to \infty$. Satisfying the condition $Q_R\big|_{y=0} = 1$ and returning to the original notation one then obtains

$$B_{1R}(\bar{k}) = 1 - \sum_{n=0}^{\infty} (\psi(n+1) + \psi(n+2) - 2\log(\bar{k}/2))\,\frac{(\bar{k}/2)^{2n+2}}{n!(n+1)!}$$

With Eq. (B.28) one then obtains

$$\bar{B}_R = \bar{B}_{1R}\,(\bar{k}) + \bar{B}_{2R}\,(\bar{k},\bar{V})$$

$$\bar{B}_R = -\,\bar{V}^{-1}(\bar{V}^2+1)^{-1/2}\cos(\bar{k}\bar{V}) + \bar{V}^{-1}(\bar{V}^2+1)^{1/2} + 1 \qquad \text{(B.58)}$$

$$+\,2(\bar{V}^2+1)^{1/2}\sum_{n=1}^{\infty}\frac{(\bar{k}/2)^{2n+2}}{n!(n+1)!}\sum_{\ell=1}^{n}\beta_{2\ell-1}\,\bar{V}^{2\ell-1}$$

$$+\,2\log((\bar{V}^2+1)^{1/2} - \bar{V})\sum_{n=0}^{\infty}\frac{(\bar{k}/2)^{2n+2}}{n!(n+1)!}$$

$$-\sum_{n=0}^{\infty}(\psi(n+1) + \psi(n+2) - 2\log(\bar{k}/2))\frac{(\bar{k}/2)^{2n+2}}{n!(n+1)!}$$

(Here the summation subscript in the second row of Eq. (B.28) has been changed by setting $j = (n+1)$. The next step would be to return to the original variables k and $\tilde{r}$, and to form $B = B_R + i\, B_i$. The first terms in these expressions then combine to form as contributions to $\bar{B}_R + i\, \bar{B}_I$.

$$-\tilde{V}^{-1}(\tilde{V}^2 + 1)^{-1/2} \exp(i\bar{k}\tilde{V})$$

or because of Eq. (B.2)

$$-\tilde{V}^{-1}(\tilde{V}^2 + \tilde{r}^2)^{-1/2} \exp(ik\tilde{V}) \qquad \text{(B.59)}$$

Within the kernel K, B appears in combination with

$$M\,\tilde{R}^{-1}(\tilde{V}^2 + \tilde{r}^2)^{-1/2} \exp(ik\tilde{V})$$

This and the expression Eq. (B.59) can be combined. We recall the following relations (Eq. (2))

$$\begin{aligned} \tilde{r} &= [(\tilde{\eta}-\tilde{y})^2 + \tilde{z}^2)]^{1/2} \\ \tilde{R} &= [(\tilde{\xi}-\tilde{x})^2 + \beta^2\, \tilde{r}^2]^{1/2} \\ \tilde{V} &= -\beta^{-2}((\tilde{\xi}-\tilde{x}) + MR) \end{aligned} \qquad \text{(B.60)}$$

Then

$$(\tilde{V}^2 + \tilde{r}^2)^{1/2} = \beta^{-2}(\tilde{R} + M(\tilde{\xi}-\tilde{x})$$

$$\begin{aligned} \log(\tilde{V}^2 + \tilde{r}^2)^{1/2} - \tilde{V}) &= \log[\beta^{-2}(\tilde{R} + M(\xi-x) + (\xi-x) + MR)] \\ &= \log[(1-M)^{-1}(\tilde{R} + \tilde{\xi}-\tilde{x})] = \log[(1+M)\tilde{r}^2(\tilde{R} - (\tilde{\xi}-\tilde{x}))^{-1}] \end{aligned} \qquad \text{(B.60a)}$$

Using these relations one obtains

$$\exp(ik\tilde{V})(\tilde{V}^2 + r^2)^{-1/2}\left[-\frac{\tilde{1}}{\tilde{V}} + \frac{M}{\tilde{R}}\right] = -\frac{1}{\tilde{R}\tilde{V}} \exp(ik\tilde{V})$$

The vanishing of the factor $(\tilde{V}^2 + \tilde{r}^2)^{-1/2}$ is a significant simplification for subsequent integrations with respect to ξ and η. The factor $\tilde{V}^{-1}$ occurs only in the first term of a development with respect to k. To display this we write

$$\exp(ik\tilde{V})(\tilde{V}^2 + \tilde{r}^2)^{-1/2}\left[\frac{1}{\tilde{V}} + \frac{M}{\tilde{R}}\right] = -\frac{1}{\tilde{R}\tilde{V}}[\exp(ik\tilde{V}) - 1] - \frac{1}{\tilde{R}\tilde{V}}$$

The term in the bracket is, of course, of $O(k)$. Combining the term $-(\tilde{R}\tilde{V})^{-1}$ with the other terms of order k^0 (in Eq. (B.58)), one obtains after introduction of the variables k and $\tilde{r}$, using Eqs. (B.60) and (B.61)

$$-\frac{1}{\tilde{R}\tilde{V}} + \frac{1}{\tilde{r}^2} + \frac{1}{\tilde{r}^2}\frac{(\tilde{V}^2 + \tilde{r}^2)^{1/2}}{\tilde{V}}$$

$$= \frac{1}{\tilde{r}^2} + \frac{1}{\tilde{r}^2\tilde{R}\tilde{V}}[-\tilde{r}^2 + \tilde{R}(\tilde{V}^2 + \tilde{r}^2)^{1/2}]$$

$$= \frac{1}{\tilde{r}^2} + \frac{1}{\beta^2\tilde{r}^2\tilde{R}\tilde{V}}[-\beta^2\tilde{r}^2 + \tilde{R}^2 + M\tilde{R}(\tilde{\xi}-\tilde{x})]$$

$$= \frac{1}{\tilde{r}^2} + \frac{1}{\beta^2\tilde{r}^2\tilde{R}\tilde{V}}[-\beta^2\tilde{r}^2(\tilde{\xi}-\tilde{x})^2 + \beta^2\tilde{r}^2 + M\tilde{R}(\tilde{\xi}-\tilde{x})]$$

Hence

$$-\frac{1}{\tilde{R}\tilde{V}} + \frac{1}{\tilde{r}^2} + \frac{1}{\tilde{r}^2}\frac{(\tilde{V}^2 + \tilde{r}^2)^{1/2}}{\tilde{V}} = \frac{1}{\tilde{r}^2}\frac{\tilde{R} - (\tilde{\xi}-\tilde{x})}{\tilde{R}} = \frac{\beta^2}{\tilde{R}(\tilde{R} + \tilde{\xi}-\tilde{x})}$$

The last formulation shows that for $\xi - x > 0$, and $\tilde{r} = 0$ (i.e., if the point (x,y) lies upstream of the point (ξ,η), there is no singularity in the flow field, and for $(\xi-x) < 0$ (i.e., if the point (x,y) lies in the wake of the point (ξ,η)), one has a singularity $\sim \tilde{r}^{-2}$. Assembling all this information one obtains,

$$M\tilde{R}^{-1}(\tilde{V}^2 + \tilde{r}^2)^{-1/2} \exp(ik\tilde{V}) + B_R(k,\tilde{V}) + i\, B_I(k,\tilde{V})$$

$$= - [\tilde{R}\tilde{V}]^{-1}(\exp(ik\tilde{V}) - 1) + \frac{1}{\tilde{r}^2} \frac{\tilde{R} - (\xi-x)}{\tilde{R}}$$

$$+ 2(\tilde{V}^2 + \tilde{r}^2)^{1/2} \sum_{n=1}^{\infty} \frac{(k/2)^{2n+2}}{n!(n+1)!} \sum_{\ell=1}^{\infty} \beta_{2\ell-1}\, \tilde{V}^{2\ell-1}\, \tilde{r}^{2(n-\ell)}$$

$$+ 2(k/2)^2[\log((k/2)((\tilde{V}^2 + \tilde{r}^2)^{1/2} - \tilde{V}))] \sum_{n=0}^{\infty} \frac{(k\tilde{r}/2)^{2n}}{n!(n+1)!}$$

$$- (k/2)^2 \sum_{n=0}^{\infty} [\psi(n+1) + \psi(n+2) - i\pi] \frac{(k\tilde{r}/2)^{2n}}{n!(n+1)!}$$

$$+ i(\tilde{V}^2 + \tilde{r}^2)^{1/2} \sum_{n=1}^{\infty} (-)^n\, c_{2n+1}\, k^{2n+1} \sum_{\ell=0}^{n-1} \beta_{2\ell}\, \tilde{V}^{2\ell}\, \tilde{r}^{2(n-\ell-1)}$$

Here the expressions containing $\tilde{V}$ can be expressed in terms of $\tilde{\xi}-\tilde{x}$ and $\tilde{R}$, by Eqs. (B.60) $\tilde{R}$ in turn is expressed by $\tilde{\xi}-\tilde{x}$ and $\tilde{r}$. The expression therefore appears in terms of the basic independent variables.

In the last minor step, we introduce the Prandtl-Glauert coordinate distortion. The purpose is to simplify the expression $\tilde{R}$, which introduces in certain integrations branch points of the integrand. We set accordingly:

$$\tilde{x} = x,\ \tilde{\xi} = \xi$$

$$\tilde{y} = \beta^{-1}y,\ \tilde{\eta} = \beta^{-1}\eta,\ \tilde{z} = \beta^{-1}z \qquad \text{(B.61)}$$

$$\tilde{r} = \beta^{-1}r,\ r = [(\eta-y)^2 + z^2]^{1/2}$$

$$\tilde{R}(\tilde{\xi}-\tilde{x},\tilde{r}) = R(\xi-x,r) = ((\xi-x)^2 + r^2)^{1/2}$$

$$\tilde{V}(\tilde{\xi}-\tilde{x},\tilde{r}) = V(\xi-x,r) = -\beta^{-2}(MR + \xi-x)$$

$$(\tilde{V}^2 + \tilde{r}^2)^{1/2} = \beta^{-2}[\tilde{R}+M(\tilde{\xi}-\tilde{x})] = \beta^{-2}[R+M(\xi-x)]$$

$$\log(\tilde{V}^2 + \tilde{r}^2)^{1/2} - \tilde{V}) = \log[(1-M)^{-1}(\tilde{R}+\tilde{\xi}-\tilde{x})) = \log[(1-M)^{-1}(R+\xi-x)]$$

$$= \log[(1-M)^{-1}\ r^2(R - (\xi-x))^{-1}] \qquad \text{(B.61)}$$

(cont'd)

The basic expression for the upwash now appears in the form

$$w(x,y,z) = (8\pi)^{-1}\frac{\partial}{\partial z}\left(z\iint \Delta p(\xi,\eta)K(k,\xi-x,r)d\xi d\eta\right) \qquad \text{(B.62)}$$

$$K = \beta^{-1}\ \exp(ik(\xi-x)(K_1 + K_2) \qquad \text{(B.63)}$$

$$K_1 = \frac{\beta^2}{r^2}\frac{R-(\xi-x)}{R} + (k^2/2)[\log(k/2)r^2(R - (\xi-x))^{-1}]\sum_0^\infty \frac{(kr/2\beta)^{2n}}{n!(n+1)!} \qquad \text{(B.64)}$$

$$K_1 = 2\beta^2 r^{-2} - \beta^2[R(R - (\xi-x))]^{-1} + (k^2/2)[\log((kr)^2/2)$$

$$- \log(k(R - (\xi-x))]\sum_0^\infty \frac{(kr/2\beta)^{2n}}{n!(n+1)!} \qquad \text{(B.64a)}$$

$$K_1 = \beta^2 R^{-1}(R + (\xi-x))^{-1} + (k^2/2)\log[(k/2)(R + (\xi-x))]\sum_0^\infty \frac{(kr/2\beta)^{2n}}{n!(n+1)!} \qquad \text{(B.64b)}$$

$$K_2 = -\ (RV)^{-1}\ \exp(ikV)\ -1) \qquad \text{(B.65)}$$

$$+ \beta^{-2}(R+M(\ -x))\ 2\sum_{n=1}^\infty \frac{(k/2)^{2n+2}}{n!(n+1)!}\sum_{\ell=1}^n \beta_{2\ell-1}\ V^{2\ell-1}(r\beta^{-1})^{2(n-\ell)}$$

$$+ i\sum_{n=1}^\infty (-)^n\ c_{2n+1}\ k^{2n+1}\sum_{\ell=0}^{n-1}\beta_{2\ell}\ V^2(r\beta^{-1})^{2(n-1-\ell)}$$

$$- (k/2)^2\sum_{n=0}^\infty [\psi(n+1) + \psi(n+2) + \log(1-M) - i\pi]\frac{[kr/(2\beta)]^{2n}}{n!(n+1)!}$$

where

$$c_{2n+1}/c_{2n-1} = -1/(4n^2 + 1),\ c_1 = 1$$

During the derivations also constants c_{2n} have been encountered, but in this formula they are directly expressed by factorials (see Eq. (B.27)). Moreover

$$\beta_{2n+2}/\beta_{2n} = -(2n+1)/(2n+2), \quad \beta_0 = 1$$

$$\beta_{2n+1}/\beta_{2n-1} = -2n/(2n+1), \quad \beta_1 = 1$$

$$\psi(n+1) + \psi(n+2) = -2\gamma + \sum_{m=1}^{n} m^{-1} + (n+1)^{-1}$$

$$\gamma = .5772156649$$

r, R, and V are defined in Eq. (B.61). Eq. (B.64a) displays the singularities which occur for $(\xi-x) < 0$ and $r \to 0$. Eq. (64b) shows that there are no singularities for $(\xi-x) > 0$. For points (x,y) close to points (ξ,η), for instance for points (x,y) within or close to a (ξ,η) element only very few terms of the infinite series are needed. Including powers up to k^2 one has

$$K_1 = \beta^2 r^{-2} R^{-1}(R - (\xi-x)) + (k^2/2)\log k[r^2(R - (\xi-x))^{-1}] \tag{B.66}$$

$$= \beta^2 R^{-1}(R + (\xi-x))^{-1} + (k^2/2)\log k(R + (\xi-x))$$

$$K_2 = -ikR^{-1} + (k^2/2)R^{-1}V \tag{B.67}$$

$$-(k/2)^2[\psi(1) + \psi(2) + 2\log 2 + \log(1-M) - i\pi]$$

APPENDIX C
SOME INDEFINITE INTEGRALS

Let

$$I_m^{-1/2} = \int w^m (w^2 + 1)^{-1/2}\, dw \tag{C.1}$$

Setting

$$I_m^{-1/2} = (w^2 + 1)^{1/2} \sum_0^{m-1} a_k^m\, w^k + c^m \int (w^2 + 1)^{-1/2}\, dw \tag{C.2}$$

one has

$$dI_m^{-1/2}/dw = (w^2 + 1)^{-1/2}[\sum_{k=0}^{m-1} a_k^m (k+1) w^{k+1} + \sum_1^{m-1} a_k^m k\, w^{k-1} + c^m] \tag{C.3}$$

We replace k in the first sum by j-1 and in the second sum by j+1. Then

$$dI_m^{-1/2}|dw = (w^2 + 1)^{-1/2}[\sum_{j=1}^{m} a_{j-1}^m\, j\, w^j + \sum_0^{m-2} a_{j+1}^m (j+1) w^j + c^m] \tag{C.4}$$

For $1 < j < m$ one obtains by comparison of the coefficients of the power w^j in Eq. (C.3) and in the derivative of Eq. (C.1)

$$j\, a_{j-1}^m + (j+1) a_{j+1}^m = 0 \tag{C.5}$$

Moreover, from the power w^0

$$a_1^m + c^m = 0 \tag{C.6}$$

and from the power w^m

$$m\, a_{m-1}^m = 1 \tag{C.7}$$

Eq. (C.4) amounts to two recurrence relations, one for even and one for odd values of m. Eq. (C.7) shows, that the subscripts of the a_j^m's will be odd for even m, and even for odd m. Consider odd values of m first and set

$$m = 2m_1 + 1$$

then from Eq. (C.7)

$$a_{2m_1}^{2m_1+1} = (2m_1 + 1)^{-1} \qquad (C.7a)$$

Let a sequence β_{2k} with $\beta_0 = 1$ satisfy the recurrence relation Eq. (C.5) namely

$$\beta_{2k+2}/\beta_{2k} = -(2k+1)/(2k+2) \qquad (C.8)$$

The β_{2k}'s can be expressed in terms of factorial for half-integral argument, but this has no practical significance, as one will even then evaluate the β_{2k}'s recursively. The first few of the β_{2k}'s are found in the following table

ℓ	β_ℓ
0	1
2	-1/2
4	3/8
6	-5/16
8	35/128

Because of Eq. (C.7a) one has

$$a_{2k}^{2m_1+1} = [(2m_1+1)\beta_{2m_1}]^{-1}\beta_{2k}$$

Therefore, from Eq. (C.2)

$$I_{2m_1+1}^{-1/2} = [(2m_1+1)\beta_{2m_1}]^{-1}(w^2 + 1)^{1/2} \sum_0^{m_1} \beta_{2k}\, w^{2k}$$

We write down some of these expressions

$$I_1^{-1/2} = (w^2 + 1)^{1/2}$$

$$I_3^{-1/2} = (w^2 + 1)^{1/2}[(1/3)w^2 - (2/3)]$$

$$I_5^{-1/2} = (w^2 + 1)^{1/2}[(1/5)w^4 - (4/15)w^2 + (8/15)]$$

$$I_7^{-1/2} = (w^2 + 1)^{1/2}[(1/7)w^6 - (6/35)w^4 + (8/35)w^2 - (16/35)]$$

For m even, we set $m = 2m_1$. Then

$$a_{2m_1-1}^{2m_1} = (2m_1)^{-1}$$

Setting $j = 2k$ in Eq. (C.5) one obtains the recurrence relation

$$\beta_{2k+1}/\beta_{2k-1} = -(2k)/(2k+1) \quad , \quad k \geq 1$$

we choose $\beta_1 = 1$. Then one obtains the following values

ℓ	β_ℓ
1	1
3	-2/3
5	8/15
7	-48/105 = -16/35

Then

$$a_{2k-1}^{2m_1} = [2m_1\beta_{2m_1-1}]^{-1}\beta_{2k-1}$$

Finally, from Eq. (C.6)

$$c^{2m_1} = -a_1^{2m_1} = -[2m_1\beta_{2m_1-1}]^{-1}$$

Thus,

$$I_{2m_1}^{-1/2} = [2m_1\beta_{2m_1-1}]^{-1}(w^2 + 1)^{1/2}\sum_{1}^{m}\beta_{2k-1}\, w^{2k-1} - [2m_1\beta_{2m_1-1}]^{-1}I_0^{-1/2} \tag{C.9}$$

$$I_0^{-1/2} = \int (w^2 + 1)^{-1/2}dw = \log[(w^2 + 1)^{1/2} + w] \tag{C.10}$$

Some such expressions are

$$I_2^{-1/2} = (w^2 + 1)^{1/2}[(1/2)w] - (1/2)\, I_0^{-1/2}$$

$$I_4^{-1/2} = (w^2 + 1)^{1/2}[(1/4)w^3 - (3/8)w] + (3/8)\, I_0^{-1/2} \tag{C.11}$$

$$I_6^{-1/2} = (w^2 + 1)^{1/2}[(1/6)w^5 - (5/24)w^3 + (5/16)w] - (5/16)\, I_0^{-1/2}$$

Expressions

$$I_m^{-3/2} = \int w^m(w^2 + 1)^{-3/2}\, dw \tag{C.12}$$

could be treated independently by a similar procedure. In the present context, one is led to more useful formulae for the upwash if one carries out an integration by parts to express $I_m^{-3/2}$ in terms of $I_m^{-1/2}$. One has

$$I_m^{-3/2} = -\, w^{m-1}(w^2 + 1)^{-1/2} + (m-1)\, I_{m-2}^{-1/2} \;,\; m \geq 1 \tag{C.13}$$

For m = 0, one obtains directly

$$I_0^{-3/2} = w(w^2 + 1)^{-1/2} \tag{C.14}$$

Also needed is the following integral

$$I = \int (w\cos\alpha - \sin\alpha)^{-1}(w^2 + 1)^{-1/2}dw$$

Setting

$$w = tg\theta \tag{C.15}$$

one obtains

$$I = \text{sign}(\cos\theta)\int[(tg\theta\cos\alpha - \sin\alpha)\cos\theta]^{-1}d\theta = \text{sign}(\cos\theta)\int\sin(\theta-\alpha)^{-1}d\theta$$

Then

$$\begin{aligned}&\int(w\cos\alpha - \sin\alpha)^{-1}(w^2 + 1)^{-1/2}dw \\ &\quad = \text{sign}(\cos\theta)\log[\sin(\theta-\alpha)(1 + \cos(\theta-\alpha))^{-1}]\end{aligned} \tag{C.16}$$

Furthermore

$$\begin{aligned}&\int(w\cos\alpha - \sin\alpha)^{-2}(w^2 + 1)^{-1/2}dw \\ &\quad = \text{sign}(\cos\theta)\int\sin(\theta-\alpha)^{-2}[\cos(\theta-\alpha)\cos\alpha - \sin(\theta-\alpha)\sin\alpha]d\theta\end{aligned}$$

Hence

$$\begin{aligned}&\int(w\cos\alpha - \sin\alpha)^{-2}(w^2 + 1)^{-1/2}dw \\ &= - \text{sign}(\cos\theta)\{[\cos\alpha \sin(\theta-\alpha)]^{-1} + \sin\alpha \log[\sin(\theta-\alpha)(1 + \cos(\theta-\alpha))^{-1}]\}\end{aligned} \tag{C.17}$$

APPENDIX D

A LIMITING PROCESS

The following expression will be discussed:

$$\lim_{z\to 0} \frac{\partial}{\partial z} \left\{ z^3 \int_{\eta_1}^{\eta_2} \frac{f_3(\eta)d\eta}{(\eta-y)^2 + z^2} \right\} = \lim_{z\to 0} \left\{ 3z^2 \int_{\eta_1}^{\eta_2} \frac{f_3(\eta)d\eta}{(\eta-y)^2 + z^2} \right.$$

$$\left. - 2z^4 \int_{\eta_1}^{\eta_2} \frac{f_3(\eta)d\eta}{[(\eta-y)^2 + z^2]^2} \right\}$$

In both integrals we replace $f_3(\eta)$ by the maximum of $|f_3(\eta)| = |f_3|max$. One has

$$\int_{\eta_1}^{\eta_2} \frac{d\eta}{(\eta-y)^2 + z^2} = z^{-1} \text{ arctg } \frac{\eta-y}{z}$$

As the arctg remains bounded even when its argument tends to infinity, (as it does for $z\to 0$) the first term is O(z) because of the factor $3z^2$ in front. Moreover

$$\int \frac{d\eta}{(\eta-y)^2 + z^2} = z^{-3}\left[\frac{1}{2} \frac{z(\eta-y)}{(\eta-y)^2 + z^2} + \text{arctg } \frac{\eta-y}{z}\right]$$

Also the second term is O(z).

APPENDIX E

THE EVALUATION OF INTEGRALS $\int q^n\phi_1 dq$, $\int q^n\phi_2 dq$, AND $\int q^n\phi_4 dq$

First we derive some auxiliary relations. Considering U and V as functions of p and q (Eqs. (164)) one has

$$\partial U/\partial q = \sin\delta$$

$$\partial V/\partial q = \cos\delta$$

Furthermore, since $R = (U^2 + V^2)^{1/2}$

$$\frac{\partial}{\partial V}\log(R-V)\Big|_{U = \text{const}} = -R^{-1}$$

$$\begin{aligned}\frac{\partial}{\partial U}\log(R-V)\Big|_{V = \text{const}} = (R-V)^{-1}UR^{-1} &= (R+V)U^{-1}R^{-1} \\ &= U^{-1} + VU^{-1}R^{-1}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial q}\log(R-V) &= -R^{-1}\cos\delta + R^{-1}VU^{-1}\sin\delta + U^{-1}\sin\delta \\ &= R^{-1}U^{-1}(-U\cos\delta + V\sin\delta) + U^{-1}\sin\delta\end{aligned}$$

Then with Eqs. (164)

$$\begin{aligned}\frac{\partial}{\partial q}\log(R-V) &= -pR^{-1}U^{-1} + U^{-1}\sin\delta \\ \frac{\partial}{\partial q}\log((R-V)|U|^{-1}) &= -pR^{-1}U^{-1}\end{aligned} \tag{E.1}$$

Similarly

$$\frac{\partial}{\partial q}\log((R-X)|Y|^{-1}) = pR^{-1}Y^{-1} \tag{E.2}$$

(Notice the difference in sign.)

Consider now

$$\int q^n \phi_1 dq = \int q^n \log(k(R-X))dq = \int q^n[\log((R-X)/Y) + \log(kY)]dq$$

With Eqs. (E.2) and Eqs. (162), viz.

$$Y = q \cos \gamma - p \sin \gamma$$

one obtains

$$\int q^n \phi_1 dq = (n+1)^{-1}\{q^{n+1}\log(k(R-X)) - p\int q^{n+1}R^{-1}Y^{-1}dq - \cos\gamma\int q^{n+1}Y^{-1}dq$$

Carrying out the division one finds

$$q^{n+1}Y^{-1} = \cos^{-1}\gamma[q^n+(p\mathrm{tg}\gamma)q^{n-1}+(p\mathrm{tg}\gamma)^2q^{n-1}\ldots(p\mathrm{tg}\gamma)^n]+(p\mathrm{tg}\gamma)^{n+1}Y^{-1}$$

There are two integrals with the factor $(p\mathrm{tg}\gamma)^{n+1}$, namely

$$-(p\mathrm{tg}\gamma)^{n+1}[p\int R^{-1}Y^{-1}dq + \cos \gamma \int Y^{-1}dq]$$

Hence with Eq. (E.2)

$$= -(p\mathrm{tg}\gamma)^{n+1}[\log((R-X)|Y|^{-1}) + \log(kY)$$

$$= -(p\mathrm{tg}\gamma)^{n+1} \log(k(R-X))$$

Therefore,

$$\int q^n \phi_1 dq = (n+1)^{-1}\{(q^{n+1} - (p\mathrm{tg}\gamma)^{n+1})\log(k(R-X)) \qquad \text{(E.3)}$$

$$- p \cos^{-1}\gamma[\int R^{-1}(q^n + (p\mathrm{tg}\gamma)q^{n-1}\ldots(p\mathrm{tg}\gamma)^n]dq$$

$$- [(n+1)^{-1}q^{n+1} + (p\mathrm{tg}\gamma)n^{-1}q^n + (p\mathrm{tg}\gamma)^2(n-1)^{-1}q^{n-1}\ldots(p\mathrm{tg}\gamma)^n q]$$

The treatment of $\int q^n \phi_2 dq = \int q^n \log(R-V)|U|^{-1})$ is nearly the same.

Beside Eqs. (E.1) one has Eq. (164), viz.

$$U = q \sin \delta + p \cos \delta$$

$$\int q^n \phi_2 dq = (n+1)^{-1} \{ q^{n+1} \log(R-V)|U|^{-1}) + p \int R^{-1} q^{n+1} (q\sin\delta + p\cos\delta)^{-1} dq \}$$

$$q^{n+1}(q\sin\delta+p\cos\delta)^{-1} = \sin^{-1}\delta[q^n + (-p\cot\delta)q^{n-1} + (-p\cot\delta)^2 q^{n-2}$$

$$+ (-\cot\delta p)^n + (-p\cot\delta)^{n+1} U^{-1}$$

Hence,

$$\int q^n \phi_2 dq = (n+1)^{-1} \{ (q^{n+1} - (-p\cot\gamma)^{n+1} \log((R-V)|U|^{-1})$$
$$+ \sin^{-1}\delta \int R^{-1} [q^n + (-p\cot\delta) q^{n-1} + (-p\cot\delta)^2 q^{n-2} + (-p\cot\delta)^n] dq \} \qquad (E.4)$$

Finally we consider

$$\int q^n \phi_4 dq = \int q^n R Y^{-1} dq = \int (p^2 q^n + q^{n+2}) Y^{-1} R^{-1} dq$$

Now

$$p^2 q^n Y^{-1} = p^2 \cos^{-1}\gamma [q^{n-1} + (ptg\gamma) q^{n-2} + (ptg\gamma)^n]^{-1} + p^2 (ptg\gamma)^n Y^{-1}$$

$$q^{n+2} Y^{-1} = \cos^{-1}\gamma [q^{n+1} + (ptg\gamma) q^n + \ldots (ptg\gamma)^{n+1}] + (ptg\gamma)^{n+2} Y^{-1}$$

One notices that terms with the same power of p can be combined. For instance

$$p^2 (ptg\gamma)^n + (ptg\gamma)^{n+2} = p^{n+2} (tg\gamma)^n \cos^{-2}\gamma \qquad (E.5)$$

but in general the result appears rather more complicated.

$$\int q^n \phi_4 dq = \cos^{-1}\gamma \int R^{-1}[q^{n+1} + (p tg\gamma q^n + (p tg\gamma)^2 q^{n-1} \ldots (p tg\gamma)^{n+1}$$

$$+ p^2 q^{n-1} \ldots . p^2 (p tg\gamma)^{n-1}] dq + p^{n+1} (tg\gamma)^n \cos^{-2}\gamma \log((R-X)/|Y|^{-1})$$

In the last term Eq. (E.2) has been applied.

APPENDIX F
LIMIT $\alpha \to \pi/2$

Only some of the expressions for the upwash written down for $\alpha \neq \pi/2$ allow one to form the limit $\alpha \to \pi/2$ directly. The others give either infinity or the difference between two quantities that tend to infinity. The results for $\alpha = \pi/2$ have been obtained by direct computation. This suffices for practical work. The limiting process $\alpha \to \pi/2$ has some value as an exercise, besides it provides a cross check of the formulae.

The type of difficulty can be seen in the following examples. Consider

$\int \frac{dx}{2x+\varepsilon}$ in the limit $\varepsilon \to 0$. One has

$$\int \frac{dx}{\varepsilon x + c} = \begin{Bmatrix} \varepsilon^{-1} \log(\varepsilon x + c) \\ x/c \end{Bmatrix} \quad \begin{matrix} \text{for } \varepsilon \neq 0 \\ \text{for } \varepsilon = 0 \end{matrix}$$

Now

$$\varepsilon^{-1} \log(\varepsilon x + c) = \varepsilon^{-1} \log c + \varepsilon^{-1} \log(1 + (\varepsilon x/c))$$

The first term can be regarded as a constant of integration, which tends infinity as ε tends to zero. Developing the logarithm with respect to ε one obtains indeed x/c.

The example shows that the idea of a development cannot always be avoided, even if one would include the specific constant of integration in the formula for the indefinite integral.

We derive a number of recurring limiting expressions. The order of the error in terms of $\Delta\alpha = \alpha - \pi/2$ is shown

$$(1 - \sin\alpha)\cos^{-2}\alpha = (1/2) + 0(\Delta\alpha)^2 \qquad (F.1)$$

$$\cos^{-1}\alpha - tg\alpha = 0(\Delta\alpha) \qquad (F.2)$$

Using Eqs. (133) one obtains

$$U = -Y+0(\Delta\alpha)$$
$$V = X+0(\Delta\alpha) \tag{F.3}$$

More specifically,

$$V-X = Y\cos\alpha-X(1-\sin\alpha) = Y\cos\alpha-(1/2)X\cos^2\alpha+0(\Delta\alpha)^3 \tag{F.4}$$

$$U+Y = X\cos\alpha+Y(1-\sin\alpha) = X\cos\alpha+(1/2)Y\cos^2\alpha+0(\Delta\alpha)^3 \tag{F.5}$$

$$U^{-1} = -Y^{-1}+U^{-1}+Y^{-1} \tag{F.6}$$

$$U^{-1} = -Y^{-1}+(U+V)U^{-1}Y^{-1} = -Y^{-1}-X\cos\alpha Y^{-2}+0(\Delta\alpha)^2 \tag{F.7}$$

$$VU^{-1} = (X\sin\alpha+Y\cos\alpha)(X\cos\alpha-Y\sin\alpha)^{-1}$$

$$= -XY^{-1}+(XY^{-1}+(X\sin\alpha+Y\cos\alpha)(X\cos\alpha-Y\sin\alpha)^{-1})$$

$$= -XY^{-1}+(X^2\cos\alpha-XY\sin\alpha+XY\sin\alpha+Y^2\cos\alpha) \tag{F.8}$$

$$VU^{-1} = -XY^{-1}-\cos\alpha R^2Y^{-2}+0(\Delta\alpha)^2 \tag{F.9}$$

$$UY^{-1} = -1+(U+Y)Y^{-1} = -1+\cos\alpha XY^{-1}+(1/2)\cos^2\alpha+0(\Delta\alpha^3)$$

$$\log((R-V)(R-X)^{-1}) = \log(1-(V-X)(R-X)^{-1})$$

$$= (-Y\cos\alpha+(1/2)X\cos^2\alpha)(R-X)^{-1} - (1/2)Y^2\cos^2\alpha(R-X)^{-2}+0(\Delta\alpha^3)$$

$$\log((R-V)(R-X)^{-1}) = -\cos\alpha(RY^{-1}+XY^{-1}) - (1/2)\cos^2\alpha[R^2Y^{-2}+RXY^{-2}]+0(\Delta\alpha)^3 \tag{F.10}$$

We revert to the original notation and write down those expressions which contain factors $tg\alpha$ or $\cos\alpha$

$$I_{00}^{0}(\alpha) = \beta^2[\mathrm{arctg}(v/u) + tg\alpha \log(k(R-X)) - \cos^{-1}\alpha \log((R-V)/U)]$$

$$I_{10}^{01} = \beta^2[-\sin\alpha \cos^{-2}\alpha\ U \log(kY) + \cos^{-2}\alpha\ U^2/y]$$

$$I_{00}^{01} = -\beta^2\cos^{-1}\alpha\ U \log(kY)$$

$$I_{10}^{02} = \beta^2\{U\ tg^2\alpha \log((R-V)/U) - \sin\alpha \cos\alpha^{-2}\log((R-X)/Y) + \cos^{-1}\alpha(R/Y)\}$$

$$I_{01}^{02} = \beta^2\{U\ tg\alpha \log((R-V)/U) - \cos^{-1}\alpha \log(R-X)/Y\}$$

Because of the definitions of X, Y, u, and v one has $\mathrm{arctg}(v/u) = -\mathrm{arctg}(X/Y) + \mathrm{const}$. This term, expressed in U and V, differs from the corresponding expression in terms of X and Y by a constant only. The other two terms of I_{00}^{0} are taken together. Here $U = \mathrm{const}$. Therefore, changing the constant of integration by $\cos^{-1}\alpha \log kU$, one considers

$$tg\alpha \log(k(R-X)) - \cos^{-1}\alpha \log(k(R-V))$$

$$= (tg\alpha - \cos^{-1}\alpha)\log(k(R-X)) - \cos^{-1}\alpha \log((R-V)/(R-X)^{-1})$$

The first term is $O(\Delta\alpha)$ because of Eq. (F.1). The second term gives $RY^{-1} + XY^{-1} + O(\Delta\alpha)$ because of Eq. (F.10).

Consider next

$$I_{10}^{01}(\alpha) + I_{10}^{02}(\alpha) = \beta^2U[\cos^{-2}\alpha\ UY^{-1} + \cos^{-1}\alpha\ RY^{-1} + tg^2\alpha \log((R-V)|U|^{-1})$$

$$- \sin\alpha \cos^{-2}\alpha \log(k(R-X))]$$

$$\cos^{-2}\alpha\ UY^{-1} = -\cos^{-2}\alpha + \cos^{-1}\alpha\ XY^{-1} + (1/2) + O(\Delta\alpha)$$

$$tg^2\alpha \log((R-V)|U|^{-1}) - \sin\alpha \cos^{-2}\alpha \log(k(R-X))$$

$$= tg^2\alpha \log((R-V)(R-X)^{-1}) - tg^2\alpha \log(k|U|)$$

$$+ (tg^2\alpha - \sin\alpha \cos^{-2}\alpha)\log(k(R-X))$$

$$= - \cos^{-1}\alpha (RY^{-1}+XY^{-1}) - (1/2)(R^2Y^{-2} + RXY^{-2})$$

$$- (1/2)\log(k(R-X)) - tg^2\alpha \log(k|U|) + O(\Delta\alpha)$$

Therefore, if one disregards constants of integrations

$$I_{10}^{0}(\pi/2) = I_{10}^{01}(\pi/2) + I_{10}^{02}(\pi/2) = -\beta^2Y[\cos^{-1}\alpha XY^{-1} + \cos^{-1}\alpha(RY^{-1} + XY^{-1})$$

$$- (1/2)(R^2Y^{-2} + RXY^{-2}) - (1/2)\log(k(R-X))]$$

$$I_{10}^{0}(\pi/2) = (\beta^2/2)[Y \log(k(R-X)) + XRY^{-1} + R^2Y^{-1}]$$

but $R^2Y^{-1} = Y + X^2Y^{-1}$ and for $\alpha = \pi/2 = Y = \text{const.}$

Therefore, by another change of the constant of integration, one obtains

$$I_{10}^{0} = (\beta^2/2)[Y \log(k(R-X) + (XR + X^2)Y^{-1})]$$

$$I_{01}^{0}(\alpha) = \beta^2U\{tg\alpha \log((R-V)|U|^{-1}) - \cos^{-1}\alpha \log(k(R-X))\}$$

$$= \beta^2U\{(tg\alpha - \cos^{-1}\alpha)\log((R-V)|U^{-1}|) + \cos^{-1}\alpha \log((R-V)(R-X)^{-1})$$

$$- \cos^{-1}\alpha \log(k|U|)\}$$

$$= \beta^2U\{O(\Delta\alpha) - RY^{-1} - XY^{-1} - \cos^{-1}\alpha \log(k|U|)\}$$

Hence

$$I_{01}^{0}(\pi/2) = \beta^2(R+X)$$

The limiting process $\alpha \to \pi/2$ is immediately obvious in I^1.., I^{21}.., and I^2_{22}.. (In other words, no expression $\cos^{-1}\alpha$ or $tg\alpha$ are encountered in these formulae.). Limiting processes are, however, needed in Q_0. One has

$$Q_0(\alpha) = (V/U)(\log(k(R-X))-1) - tg\alpha \log(k(R-X))$$

$$+ \cos^{-1}\alpha \log((R-V)|U|^{-1})$$

Here

$$-tg\alpha \log(k(R-X)) + \cos^{-1}\alpha \log((R-V)|U|)^{-1})$$

$$= \cos^{-1}\alpha \log((R-V)(R-X)^{-1}) - \cos^{-1}\alpha(\log(k|U|) + O(\Delta\alpha)$$

$$= -RY^{-1} + Y^{-1} - \cos^{-1}\alpha \log(k|U|) + O(\Delta\alpha)$$

Therefore, disregarding constants

$$Q_0(\pi/2) = -XY^{-1}\log(k(R-X)) - RY^{-1}$$

$$Q_1 = (1/2)\{V^2U^{-2}\log(k(R-X))-tg^2\alpha\log(k(R-X)+\sin\alpha\cos^{-2}\alpha\log(R-V)|U|^{-1})$$

$$- (1/2)V^2U^{-2}-tg\alpha VU^{-1} - \cos^{-1}\alpha\ RU^{-1}\}$$

$$= (1/2)\{X^2Y^{-2}(\log(k(R-X)) - (1/2)) + tg^2\alpha \log((R-V)(R-X)^{-1})$$

$$+ (\sin\alpha \cos^{-2}\alpha - tg^2\alpha)\log(k(R-V)) - \sin\alpha \cos^{-2}\alpha \log(k|U|)$$

$$- tg\alpha(-XY^{-1} - \cos\alpha\ R^2Y^{-2}) + \cos^{-1}\alpha\ R(Y^{-1} + \cos\alpha\ XY^{-2}) + O(\Delta\alpha)\}$$

$$= (1/2)\{X^2Y^{-2}\log(k(R-X)) - (1/2)) + tg^2\alpha(-\cos\alpha(RY^{-1} + XY^{-1})$$

$$- (1/2)\cos^2\alpha(R^2Y^{-2} + RXY^{-2})] + (1/2)\log(k(R-X)) + \cos^{-1}\alpha\ XY^{-1}$$

$$+ R^2Y^{-2} + \cos^{-1}\alpha\ RY^{-1} + RXY^{-2} - \sin\alpha \cos^{-2}\alpha \log(k|U|) + O(\Delta\alpha)\}$$

$$= (1/2)\{X^2Y^{-2}\log(k(R-X)) - (1/2) + (1/2)\log(k(R-X))$$

$$+ (1/2)R^2Y^{-2} + (1/2)RXY^{-2}\} - \sin\alpha\,\cos^{-2}\alpha\,\log(k|U|) + O(\Delta\alpha)\}$$

but $(1/2)R^2Y^{-2} = (1/2) + (1/2)X^2Y^{-2}$.

Therefore,

$$Q_1 = (1/2)\{\log(k(R-X))(X^2Y^{-2} + (1/2)) + (1/2)RXY^{-2} + f(\alpha)\}$$

With these expressions the limiting process in I^{23}.. can be carried out.

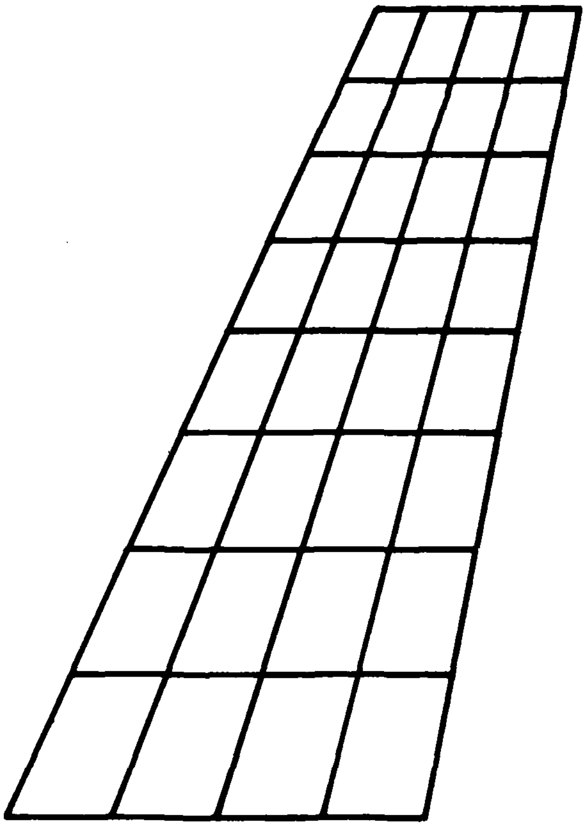


Figure 1. Half-wing with trapezoidal elements.

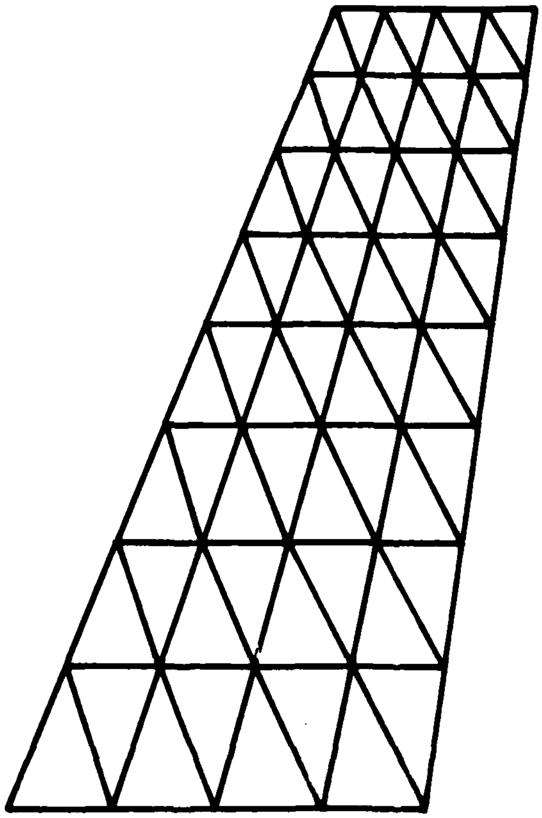
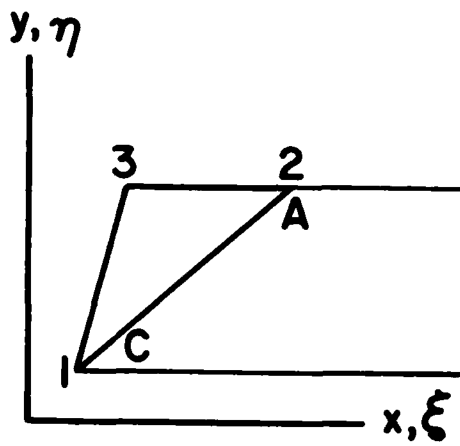
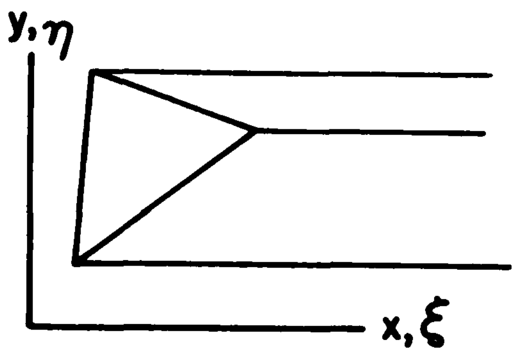


Figure 2. Half-wing with triangular elements.



(3a)



(3b)

Figure 3. Triangular elements with lines of the wake along which singularities occur. (Along line AB of Figure 3a there is a singularity as $(y-\eta)^{-1}$, along all other lines as $\log|y-\eta|$.)

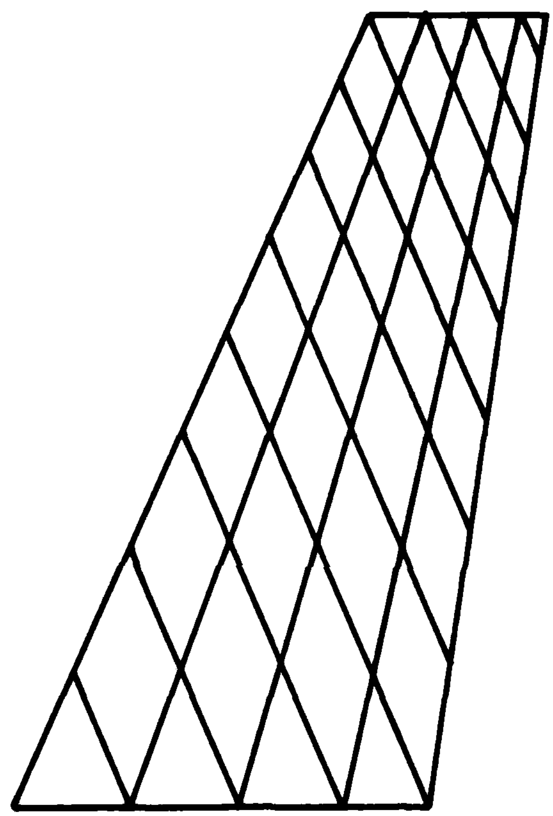
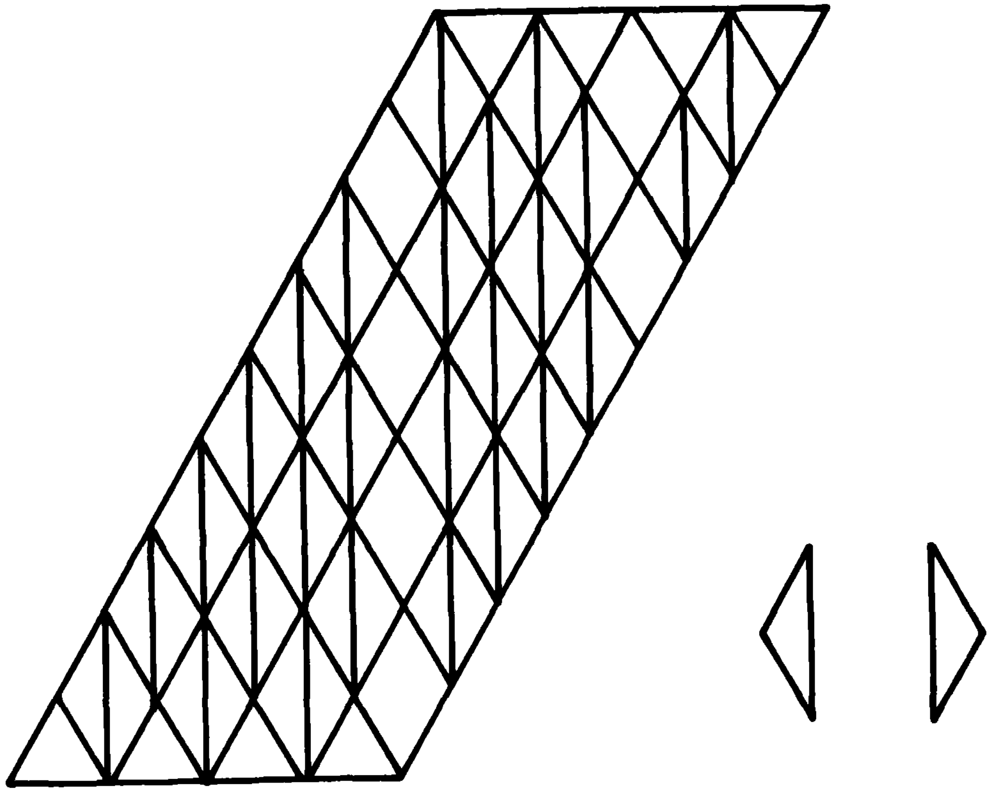


Figure 4. Subdivision of the wing surface in which no element side is a line η = const.



(5a)

Figure 5. Wing with leading and trailing edges parallel to each other, and elements given by congruent triangles. Figure 5a shows the two types of elements encountered.

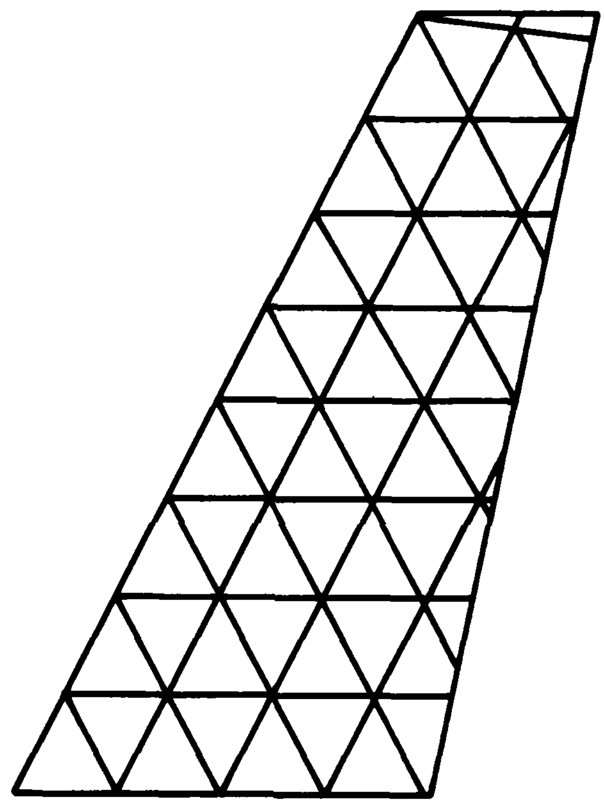


Figure 6. Wing with no parallel leading and trailing edges, and the same kind of elements as in Figure 5. Exceptional elements appear at the trailing edge.

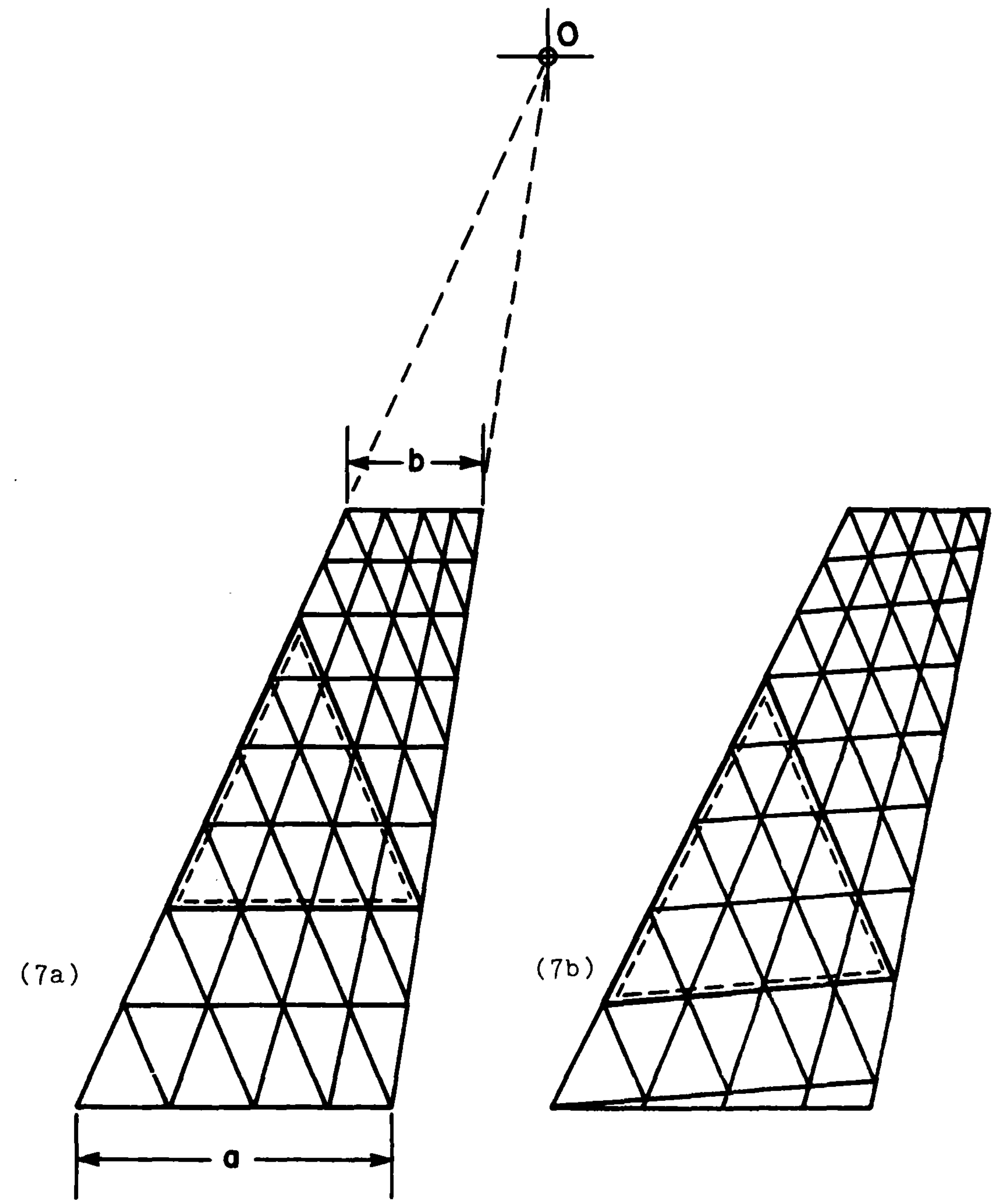


Figure 7. Two subdivisions of the wing surface into elements which are self-similar. In Figure 7a element boundaries are lines η = const. In Figure 7b element boundaries of adjacent elements lie on one straight line.

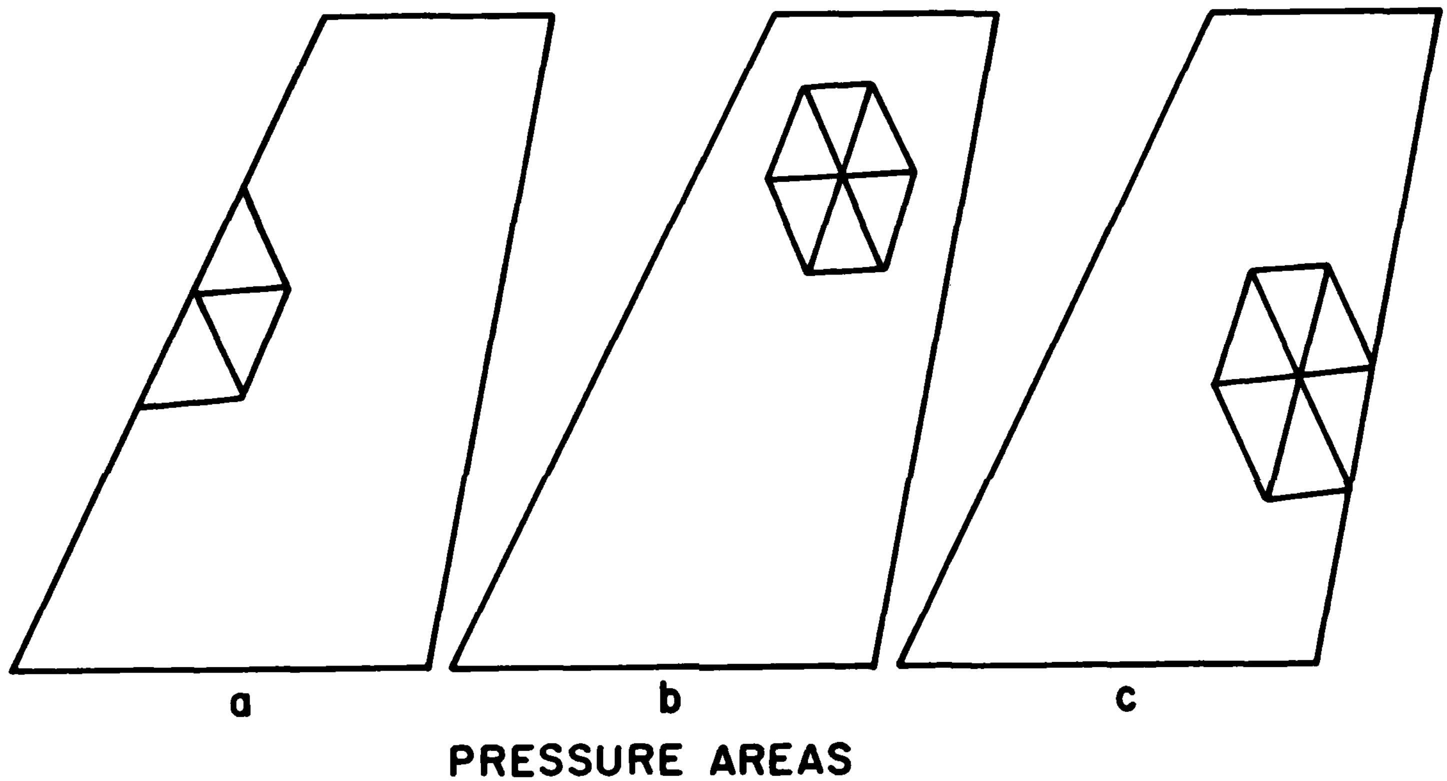


Figure 8. Pressure and upwash areas, (a) for points of the leading edge, (b) for points in the interior, and (c) for points next to the trailing edge.

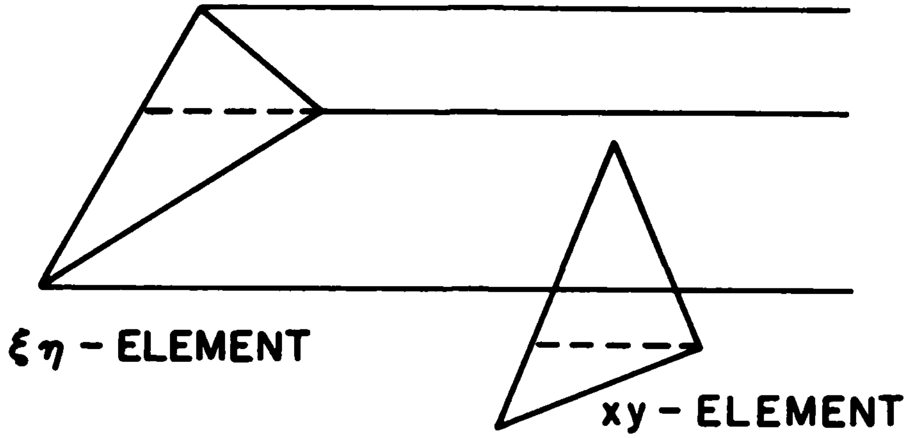


Figure 9. Subdivision of $\xi\eta$- and xy-elements.

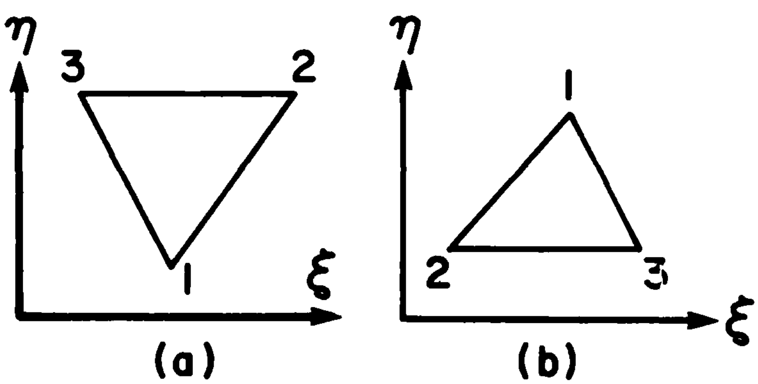


Figure 10. Numbering of the corners in an $\xi\eta$-element with one side parallel to the η-axis.

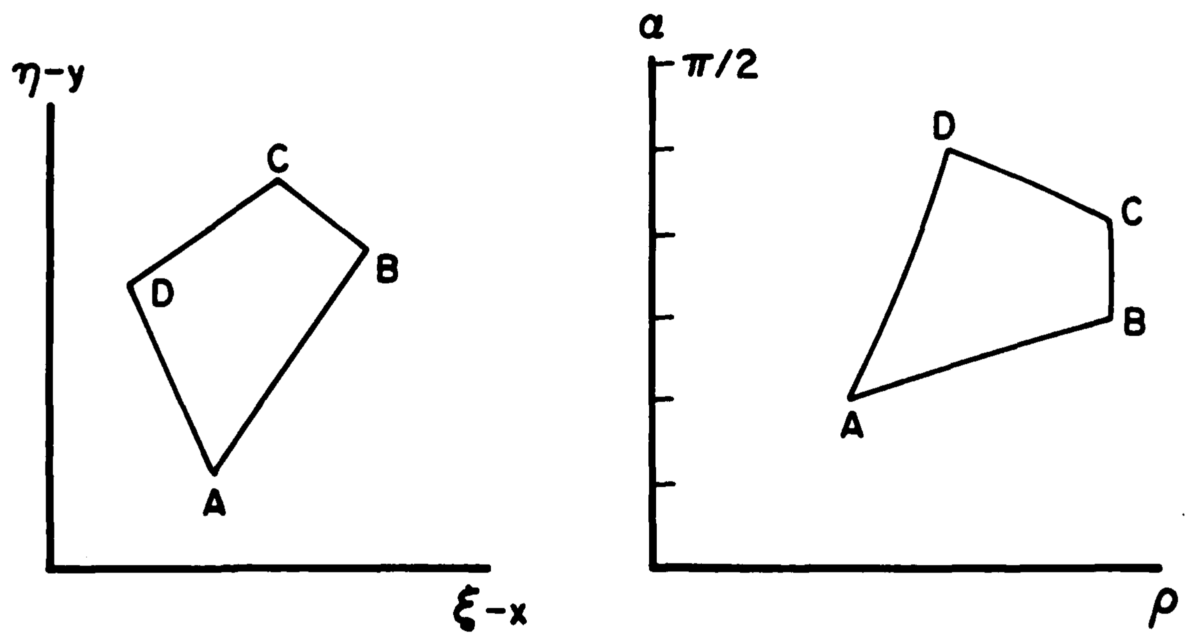
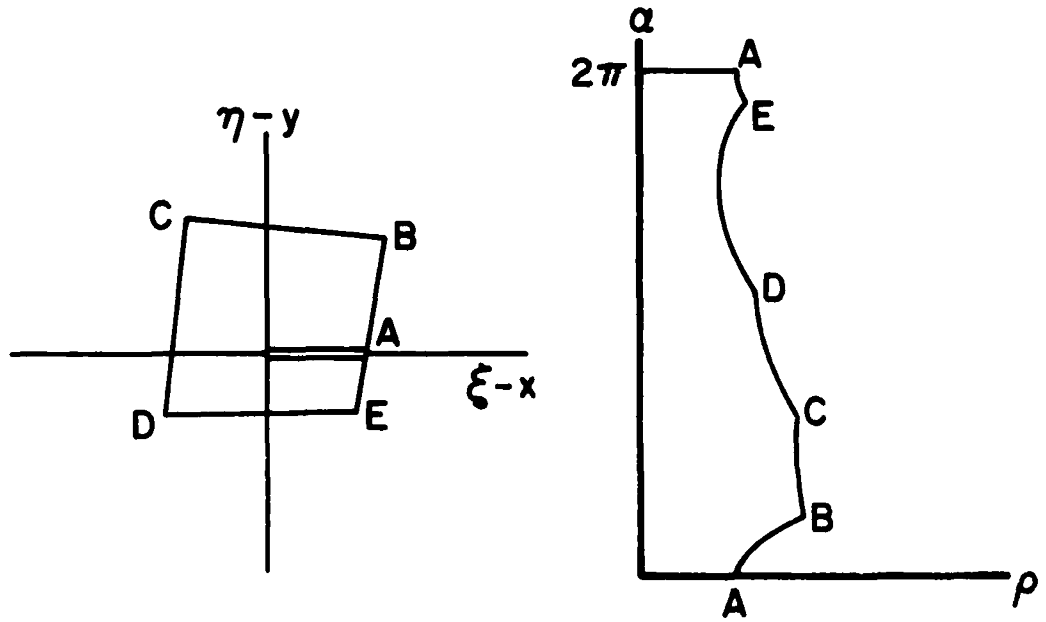


Figure 11. Map of an element from the ξ,η plane to a ρ,α plane, if the point ξ-x = 0, η-y = 0 lies outside the element.



Fiugre 12. Map of an element from the ξ,η plane to a ρ,α plane, if the point ξ-x = 0, η-y = 0 lies inside the element.

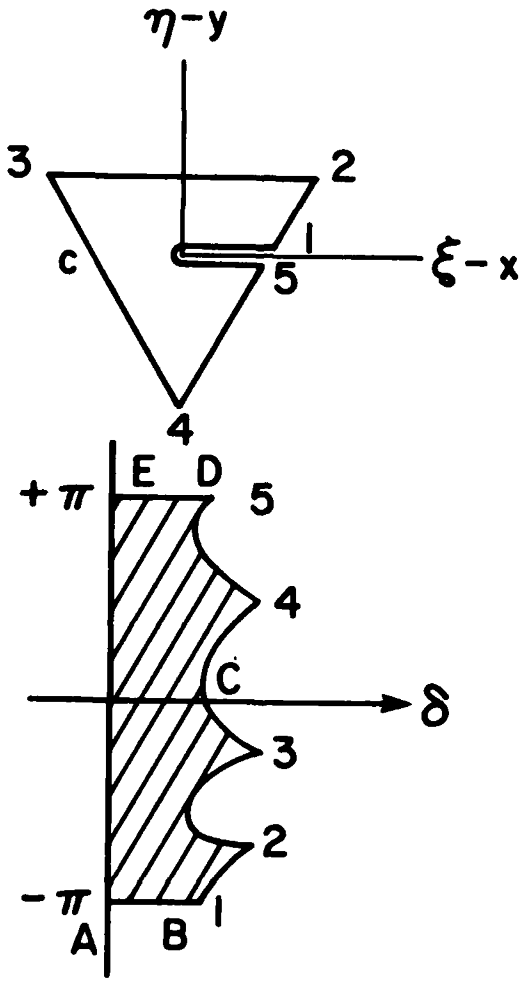


Figure 13. Map of a triangular element from the ξ,η plane to a ρ,α plane, if the point ξ-x = 0, η-y = 0 lies inside the element.

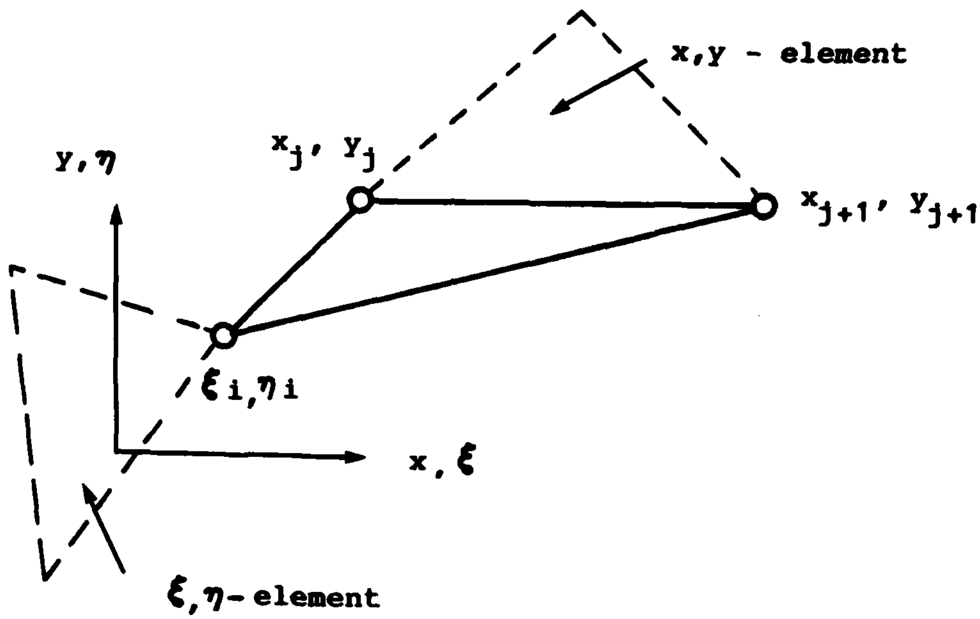


Figure 14. Choice of the sign for the contribution of the triangle ξ_i, η_i, x_{j+1}, y_{j+1}, x_j, y_j.

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